

DEFORMATIONS OF 3-ALGEBRAS

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ABSTRACT. We phrase deformations of n -Leibniz algebras in terms of the cohomology theory of the associated Leibniz algebra. We do the same for n -Lie algebras and for the metric versions of n -Leibniz and n -Lie algebras. We place particular emphasis on the case of $n = 3$ and explore the deformations of 3-algebras of relevance to three-dimensional superconformal Chern–Simons theories with matter.

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1. INTRODUCTION

It is often said that one can learn a lot about a mathematical object by studying how it behaves under deformations. In this paper we will study the deformations of certain algebraic structures which have appeared recently in the Bagger–Lambert–Gustavsson theory of multiple M2-branes [1–3]. More generally, they underlie certain three-dimensional superconformal Chern–Simons theories coupled to matter, where the matter fields take values in a metric 3-Leibniz algebra (see below for a precise definition). Not every 3-Leibniz algebra is associated

to one such theory, but those which can be constructed from a metric Lie algebra and a faithful unitary representation [4] via a construction originally due to Faulkner [5]. A special class of such algebras are the metric 3-Lie algebras, which go back to the work of Filippov

Let V be a real n -Lie algebra [6]; that is, a real vector space and an alternating multilinear map $V^n \rightarrow V$ denoted by

$$(x_1, \dots, x_n) \mapsto [x_1, \dots, x_n] \quad (1)$$

satisfying a *fundamental identity* which says that the endomorphisms of V defined by $y \mapsto [x_1, \dots, x_{n-1}, y]$ are derivations for all $x_i \in V$. This suggests defining a linear map $D : \Lambda^{n-1}V \rightarrow \text{End } V$ by

$$D(x_1 \wedge \dots \wedge x_{n-1}) \cdot y = [x_1, \dots, x_{n-1}, y] \quad (2)$$

and extending it linearly to all of $\Lambda^{n-1}V$. In terms of this map, the fundamental identity can be written as a simple relation in $\text{End } V$.

Lemma 1. *The fundamental identity is equivalent to*

$$[D(X), D(Y)] = D(D(X) \cdot Y) , \quad (3)$$

for all $X, Y \in \Lambda^{n-1}V$, where the \cdot in the right-hand side is the natural action of $\text{End } V$ on $\Lambda^{n-1}V$ and the bracket on the left-hand side is the commutator in $\text{End } V$.

Proof. Indeed, let $X = x_1 \wedge \dots \wedge x_{n-1}$ and $Y = y_1 \wedge \dots \wedge y_{n-1}$ be monomials. Then

$$\begin{aligned} D(X) \cdot Y &= D(x_1 \wedge \dots \wedge x_{n-1}) \cdot (y_1 \wedge \dots \wedge y_{n-1}) \\ &= \sum_{i=1}^{n-1} y_1 \wedge \dots \wedge [x_1, \dots, x_{n-1}, y_i] \wedge \dots \wedge y_{n-1} , \end{aligned}$$

whence applying the fundamental identity to $y_n \in V$, we find

$$[D(X), D(Y)] \cdot y_n = [x_1, \dots, x_{n-1}, [y_1, \dots, y_{n-1}, y_n]] - [y_1, \dots, y_{n-1}, [x_1, \dots, x_{n-1}, y_n]]$$

whereas

$$\begin{aligned} D(D(X) \cdot Y) \cdot y_n &= \sum_{i=1}^{n-1} D(y_1 \wedge \dots \wedge [x_1, \dots, x_{n-1}, y_i] \wedge \dots \wedge y_{n-1}) \cdot y_n \\ &= \sum_{i=1}^{n-1} [y_1, \dots, [x_1, \dots, x_{n-1}, y_i], \dots, y_{n-1}, y_n] . \end{aligned}$$

Into the fundamental identity and rearranging terms we obtain

$$[x_1, \dots, x_{n-1}, [y_1, \dots, y_{n-1}, y_n]] = \sum_{i=1}^n [y_1, \dots, [x_1, \dots, x_{n-1}, y_i], \dots, y_{n-1}, y_n] , \quad (4)$$

which says precisely that $D(x_1 \wedge \dots \wedge x_{n-1})$ is a derivation over the bracket. \square

The fundamental identity in the form (3) says that the image of D is a Lie subalgebra of $\mathfrak{gl}(V)$. At first sight, this is unexpected, because D is only a linear map and hence there is no right to expect that its image should be a Lie subalgebra. A natural explanation for this fact would be that D is a Lie algebra morphism, but this would require $\Lambda^{n-1}V$ to possess the structure of a Lie algebra. Taking this seriously and glancing at equation (3), we would be tempted to define the bracket in $\Lambda^{n-1}V$ by

$$[X, Y] := D(X) \cdot Y , \quad (5)$$

in such a way that the fundamental identity (3) becomes

$$[D(X), D(Y)] = D([X, Y]) . \quad (6)$$

Although the bracket defined by (5) is not skewsymmetric, it does however obey a version of the Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] , \quad (7)$$

for all $X, Y, Z \in \Lambda^{n-1}V$. (We will prove this later in a slightly more general context.) It turns out that this makes $\Lambda^{n-1}V$ into a (*left*) *Leibniz algebra* (see below for a precise definition) and then the fundamental identity in the form (6) makes D into a (*left*) Leibniz algebra morphism, whence explaining satisfactorily the fact that its image is a Leibniz (and hence Lie) subalgebra of $\mathfrak{gl}(V)$. (This observation seems to have been made by Daletskii in the first instance.)

It turns out that this is the correct framework in which to discuss the cohomology of n -Lie algebras. In particular we will show that deformations of an n -Lie algebra are governed by certain cohomology groups of the associated Leibniz algebra $\Lambda^{n-1}V$. Leibniz algebras were introduced by Loday in [7] and the cohomology was discussed by Cuvier in [8, 9] for the special case of symmetric representations (see below for a precise definition) and by Loday and Pirashvili in [10] in general. The question of deformations of n -Lie algebras (motivated by the quantisation problem for Nambu mechanics [11]) was tackled successfully by Gautheron in [12], after an initial attempt by Takhtajan in [13]. Daletskii and Takhtajan in [14] rewrote Gautheron's result in terms of Leibniz cohomology. One purpose of this paper is to obtain a clearer statement of which cohomology groups control the deformations of an n -Leibniz algebra, of which an n -Lie algebra is a special case.

This paper is organised as follows. The first two sections are mostly a review of known results, collected here under one roof. In Section 2 we recall the definition of a Leibniz algebra and its associated Lie algebra. We introduce the notion of a representation and in particular the special cases of symmetric and antisymmetric representations. We recall the definition of the universal enveloping algebra of a Leibniz algebra and of the differential complex computing the cohomology $HL^*(L; M)$ of a (*left*) Leibniz algebra L with values in a representation M . In Section 3 we introduce the notion of an n -Leibniz algebra and describe a functor which assigns a (*metric*) Leibniz algebra canonically to each such (*metric*) n -Leibniz algebra. In Section 4 we show that infinitesimal deformations of V (in the sense of Gerstenhaber [15]) are classified by $HL^1(L; \text{End } V)$, where $\text{End } V$ is a *nonsymmetric* representation of L , whereas the obstructions to integrating an infinitesimal deformation live in $HL^2(L; \text{End } V)$. In order to arrive at this result we endow $CL^\bullet(L; \text{End } V)$ with a graded Lie algebra structure as in the Nijenhuis–Richardson theory of Lie algebra deformations [16]. We also discuss the deformation theory of n -Lie algebras and of *metric* n -Leibniz and n -Lie algebras. In Section 5 we specialise to the 3-Leibniz algebras of interest in the context of three-dimensional superconformal Chern–Simons theories. As shown in [4], these are in one-to-one correspondence with pairs consisting of a metric Lie algebra and a faithful orthogonal representation, via a construction originally due to Faulkner [5]. We discuss the relevant Leibniz algebra in the general Faulkner construction and set up the deformation theory in the special case of orthogonal representations. We show that the unique simple euclidean 3-Lie algebra is rigid as a 3-Lie algebra, but admits a one-parameter family of nontrivial deformations as a 3-Leibniz algebra, which we interpret in terms of deformations of its Faulkner data. Finally we discuss the deformation problem for general Faulkner data and show that if the metric Lie algebra in the Faulkner construction is semisimple, then the deformations all correspond to rescaling the Killing forms in each of its simple ideals.

2. LEIBNIZ ALGEBRAS, THEIR REPRESENTATIONS AND THEIR COHOMOLOGY

In this section we recall some definitions in the theory of Leibniz algebras and introduce the notion of representation, universal enveloping algebra and cohomology with values in a representation. The treatment follows [10].

2.1. Basic definitions. A **(left) Leibniz algebra** is a vector space L together with a bilinear map $L^2 \rightarrow L$, $(X, Y) \mapsto [X, Y]$ satisfying the **(left) Leibniz identity**:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] , \quad \text{for all } X, Y, Z \in L. \quad (8)$$

If the bracket were skewsymmetric, so that $[X, Y] = -[Y, X]$, then the Leibniz identity would become the Jacobi identity, making L into a Lie algebra. In a sense, Leibniz algebras are non-commutative versions of Lie algebras. Because of this noncommutativity one has to distinguish between left and right Leibniz algebras. Some of the relevant literature (e.g., [9, 10]) considers right Leibniz algebras, whereas in the present context the natural notion is that of a left Leibniz algebra defined above. Care must be exercised in translating from one to the other. All our Leibniz algebras will be left Leibniz algebras unless otherwise stated.

Let $K \subset L$ denote the subspace of L spanned by $\{[X, X] | X \in L\}$.

Lemma 2. *If $Z \in K$, then $[Z, X] = 0$ and $[X, Z] \in K$, for all $X \in L$.*

Proof. This follows from the Leibniz identity (8). Indeed,

$$[[X, X], Y] = [X, [X, Y]] - [X, [X, Y]] = 0 , \quad (9)$$

and

$$[Y, [X, X]] = [[Y, X], X] + [X, [Y, X]] , \quad (10)$$

but we notice that any element of the form $[X, Y] + [Y, X]$ belongs to K , since

$$[X, Y] + [Y, X] = \frac{1}{2}[X + Y, X + Y] - \frac{1}{2}[X - Y, X - Y] . \quad (11)$$

□

In other words, K is an ideal of L and the quotient $\mathfrak{g}_L := L/K$ is therefore a Leibniz algebra whose bracket is skewsymmetric by construction. This makes \mathfrak{g}_L into a Lie algebra known as the Lie algebra associated to the Leibniz algebra L . If $X \in L$, we will let $\overline{X} \in \mathfrak{g}_L$ denote the corresponding element. If \mathfrak{g} is a Lie algebra, thought of as a Leibniz algebra, then any Leibniz algebra morphism $L \rightarrow \mathfrak{g}$ factors through a unique Lie algebra morphism $\mathfrak{g}_L \rightarrow \mathfrak{g}$. In this sense, \mathfrak{g}_L is universal for Leibniz morphisms to Lie algebras.

2.2. Representations and the universal enveloping algebra. In this section we translate results from [10] from right to left Leibniz algebras.

An **abelian extension** of a Leibniz algebra L is an exact sequence

$$0 \longrightarrow M \longrightarrow L' \longrightarrow L \longrightarrow 0 \quad (12)$$

of Leibniz algebras where $[M, M] = 0$. This means that M admits two actions of L : a left action $L \times M \rightarrow M$ denoted $[X, m]$ and a right action $M \times L \rightarrow M$, denoted $[m, X]$ for $X \in L$ and $m \in M$. These actions are given by the bracket of L' after choosing *any* vector space section $L \rightarrow L'$ and as a result satisfy three compatibility conditions coming from the Leibniz identity in L' applied to two elements of L and one of M in different positions, namely:

$$[[X, Y], m] = [X, [Y, m]] - [Y, [X, m]] \quad (13)$$

$$[[X, m], Y] = [X, [m, Y]] - [m, [X, Y]] \quad (14)$$

$$[[m, X], Y] = [m, [X, Y]] - [X, [m, Y]] , \quad (15)$$

for all $X, Y \in L$ and $m \in M$. Notice that adding the last two equations, we have

$$[[X, m], Y] + [[m, X], Y] = 0 . \quad (16)$$

We say that a vector space M is a **representation** of the Leibniz algebra L if M admits two actions of L , on the left and on the right, obeying equations (13), (14) and (15). A vector subspace $N \subset M$ is a **subrepresentation** if it is closed under both the left and right actions of L . A representation M is said to be **symmetric** if $[X, m] + [m, X] = 0$ for all $X \in L$ and $m \in M$. In this case, equations (14) and (15) are equivalent to equation (13), which we may take as the defining condition for a symmetric representation. Similarly, a representation is said to be **antisymmetric** if $[m, X] = 0$. In this case, the only nontrivial condition is again equation (13).

As for a Lie algebra, a Leibniz algebra has a universal enveloping algebra: an associative algebra such that its (left) modules are in one-to-one correspondence with the representations of the Leibniz algebra. Because representations of a Leibniz algebra consists of left and right actions, the universal enveloping algebra of a Leibniz algebra L is a quotient of the tensor algebra of $L \oplus L$. For $X \in L$, we let $r_X = (0, X)$ and $\ell_X = (X, 0)$ denote the corresponding elements in $L \oplus L$. The **universal enveloping algebra** $UL(L)$ of L is the quotient of the tensor algebra $T(L \oplus L)$ by the two-sided ideal I generated by the following elements

$$\ell_{[X, Y]} - \ell_X \otimes \ell_Y + \ell_Y \otimes \ell_X \quad r_{[X, Y]} - \ell_X \otimes r_Y - r_Y \otimes \ell_X \quad r_Y \otimes (\ell_X + r_X) , \quad (17)$$

for all $X, Y \in L$.

Proposition 3. *There is a categorical equivalence between representations of L and left modules of $UL(L)$.*

Proof. If M is a representation of L we define

$$\ell_X m := [X, m] \quad \text{and} \quad r_X m := [m, X] \quad (18)$$

for all $X \in L$ and $m \in M$. We extend it to all of $T(L \oplus L)$ by linearity and composition. The conditions (13), (14) and (15) are such that the ideal I acts trivially, whence the action of $T(L \oplus L)$ induces an action of $UL(L)$. Conversely, restricting the action of $UL(L)$ to $L \oplus L$ gives the actions of L on M which satisfy conditions (13), (14) and (15) by virtue of the relations in the ideal I . \square

We can now define a **corepresentation** of a Leibniz algebra, as a right module of its universal enveloping algebra. Unlike with Lie algebras, they are now not simply given by changing the sign of the actions because changing the sign is *not* an antiautomorphism of the universal enveloping algebra. Instead we have the following

Proposition 4. *The endomorphism of $UL(L)$, defined on generators by*

$$\ell_X \mapsto -\ell_X \quad \text{and} \quad r_X \mapsto \ell_X + r_X \quad (19)$$

extends to an antiautomorphism of $UL(L)$.

Proof. It is just a matter of showing that the above map preserves the ideal I . In detail, for the first relation $R_1(X, Y)$ in (17), we find

$$R_1(X, Y) \mapsto -\ell_{[X, Y]} - (-\ell_Y) \otimes (-\ell_X) + (-\ell_X) \otimes (-\ell_Y) = -R_1(X, Y) ; \quad (20)$$

whereas for the second relation $R_2(X, Y)$ we find

$$R_2(X, Y) \mapsto r_{[X, Y]} + \ell_{[X, Y]} - (\ell_Y + r_Y) \otimes (-\ell_X) + (-\ell_X) \otimes (\ell_Y + r_Y) = R_1(X, Y) + R_2(X, Y) , \quad (21)$$

and for the last relation $R_3(X, Y)$ we find

$$R_3(X, Y) \mapsto (-\ell_X + \ell_X + r_X) \otimes (\ell_Y + r_Y) = R_3(Y, X) . \quad (22)$$

One can check that restricted to the ideal I , this antiautomorphism has order 2. \square

Unlike Lie algebras, for which homology and cohomology both can take values in representations, for a Leibniz algebra homology takes values in a corepresentation and cohomology takes values in a representation, somewhat confusingly. We will not use homology in this paper, only cohomology with values in a representation, to which we now turn.

2.3. Cohomology. Let L be a Leibniz algebra and M a representation. On the graded space $CL^\bullet(L; M) = \bigoplus_{p \geq 0} CL^p(L; M)$, with

$$CL^p(L; M) := \text{Hom}(L^{\otimes p}, M) \quad (23)$$

the space of p -linear maps from L to M , we define a differential $d : CL^p(L; M) \rightarrow CL^{p+1}(L; M)$ by the following formula

$$\begin{aligned} (d\varphi)(X_1, \dots, X_{p+1}) &= \sum_{i=1}^p (-1)^{i-1} [X_i, \varphi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1})] \\ &\quad + (-1)^{p+1} [\varphi(X_1, \dots, X_p), X_{p+1}] \\ &\quad + \sum_{1 \leq i < j \leq p+1} (-1)^i \varphi(X_1, \dots, \widehat{X_i}, \dots, [X_i, X_j], \dots, X_{p+1}) , \end{aligned} \quad (24)$$

where $\varphi \in CL^p(L; M)$, $X_i \in L$ and where $\widehat{}$ denotes omission. Notice that we have both the left and right actions of L on M in the expression for the differential. It follows from the explicit expression of the differential that if $N \subset M$ is a subrepresentation, then $CL^\bullet(L; N)$ is a subcomplex.

In the special case of M being a symmetric representation, the differential takes a somewhat simpler form

$$\begin{aligned} (d\varphi)(X_1, \dots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} [X_i, \varphi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1})] \\ &\quad + \sum_{1 \leq i < j \leq p+1} (-1)^i \varphi(X_1, \dots, \widehat{X_i}, \dots, [X_i, X_j], \dots, X_{p+1}) , \end{aligned} \quad (25)$$

reminiscent of the Chevalley–Eilenberg differential computing Lie algebra cohomology.

Let us write the first few differentials in the general case. If $m \in M = CL^0(L; M)$, then

$$dm(X) = -[m, X] . \quad (26)$$

If $\theta : L \rightarrow M$, then $d\theta : L \otimes L \rightarrow M$ is defined by

$$d\theta(X, Y) = [X, \theta(Y)] + [\theta(X), Y] - \theta([X, Y]) . \quad (27)$$

Finally, if $\varphi : L \otimes L \rightarrow M$, then $d\varphi : L^{\otimes 3} \rightarrow M$ is defined by

$$\begin{aligned} d\varphi(X, Y, Z) &= [X, \varphi(Y, Z)] - [Y, \varphi(X, Z)] - [\varphi(X, Y), Z] \\ &\quad + \varphi(X, [Y, Z]) - \varphi(Y, [X, Z]) - \varphi([X, Y], Z) . \end{aligned} \quad (28)$$

One can check that $d^2 = 0$ precisely because L is a Leibniz algebra and M is a representation [10]. Let us illustrate this for $p = 0$ and $p = 1$. Indeed, if $m \in M = CL^0(L; M)$, then

$$d^2 m(X, Y) = [X, dm(Y)] + [dm(X), Y] - dm([X, Y]) \quad \text{by equation (27)}$$

$$\begin{aligned}
&= -[X, [m, Y]] - [[m, X], Y] + [m, [X, Y]] && \text{by equation (26)} \\
&= 0 . && \text{by equation (15)}
\end{aligned}$$

Similarly, if $\theta \in CL^1(L; M)$, then

$$\begin{aligned}
d^2\theta(X, Y, Z) &= [X, d\theta(Y, Z)] - [Y, d\theta(X, Z)] - [d\theta(X, Y), Z] \\
&\quad + d\theta(X, [Y, Z]) - d\theta(Y, [X, Z]) - d\theta([X, Y], Z) && \text{by equation (28)} \\
&= [X, [Y, \theta(Z)] + [\theta(Y), Z] - \theta([Y, Z])] \\
&\quad - [Y, [X, \theta(Z)] + [\theta(X), Z] - \theta([X, Z])] \\
&\quad - [[X, \theta(Y)] + [\theta(X), Y] - \theta([X, Y]), Z] \\
&\quad + [X, \theta([Y, Z])] + [\theta(X), [Y, Z]] - \theta([X, [Y, Z]]) \\
&\quad - [Y, \theta([X, Z])] - [\theta(Y), [X, Z]] + \theta([Y, [X, Z]]) \\
&\quad - [[X, Y], \theta(Z)] - [\theta([X, Y]), Z] + \theta([X, Y], Z) && \text{by equation (27)}
\end{aligned}$$

which rearranges into

$$\begin{aligned}
&= \theta([X, Y], Z) - [X, [Y, Z]] + [Y, [X, Z]] \\
&\quad + [X, [Y, \theta(Z)]] - [Y, [X, \theta(Z)]] - [[X, Y], \theta(Z)] \\
&\quad + [X, [\theta(Y), Z]] - [[X, \theta(Y)], Z] - [\theta(Y), [X, Z]] \\
&\quad - [Y, [\theta(X), Z]] - [[\theta(X), Y], Z] + [\theta(X), [Y, Z]] ,
\end{aligned}$$

and the last four lines vanish because of equations (7), (13), (14) and (15), respectively.

3. n -LEIBNIZ ALGEBRAS AND THEIR ASSOCIATED LEIBNIZ ALGEBRAS

In this section we exhibit a functor from the category of (metric) n -Leibniz algebras to the category of (metric) Leibniz algebras.

3.1. From n -Leibniz algebras to Leibniz algebras. Let V be a (left) n -Leibniz algebra; that is, V is a vector space with a multilinear bracket $V^n \rightarrow V$, $(x_1, \dots, x_n) \mapsto [x_1, \dots, x_n]$ obeying the fundamental identity

$$[D(X), D(Y)] \cdot z = D(D(X) \cdot Y) \cdot z , \quad \text{for all } X, Y \in V^{\otimes(n-1)} \text{ and } z \in V, \quad (29)$$

where $D : V^{\otimes(n-1)} \rightarrow \text{End } V$ is defined on monomials by

$$D(x_1 \otimes \dots \otimes x_{n-1}) \cdot z = [x_1, \dots, x_{n-1}, z] , \quad (30)$$

and extended by linearity to all of $V^{\otimes(n-1)}$. Because equation (29) is true for all z , it is true abstractly as endomorphisms, whence it is formally identical to (3), but with $X, Y \in V^{\otimes(n-1)}$. On $V^{\otimes(n-1)}$ we define the following bracket

$$[X, Y] := D(X) \cdot Y . \quad (31)$$

Proposition 5. *The above bracket turns $V^{\otimes(n-1)}$ into a Leibniz algebra.*

Proof. Indeed, the Leibniz identity (7) reads

$$D(X) \cdot (D(Y) \cdot Z) = D(D(X) \cdot Y) \cdot Z + D(Y) \cdot (D(X) \cdot Z) , \quad (32)$$

or equivalently

$$[D(X), D(Y)] \cdot Z = D(D(X) \cdot Y) \cdot Z , \quad (33)$$

which follows from the fundamental identity (3). \square

We will call $V^{\otimes(n-1)}$ the **Leibniz algebra associated to the n -Leibniz algebra V** and from now on we will denote it $L(V)$, or simply L if V is implicit.

The map $V \mapsto L(V)$ extends to a functor from the category of n -Leibniz algebras to the category of Leibniz algebras, by sending a morphism $\varphi : V \rightarrow W$ of n -Leibniz algebras, to $L\varphi : L(V) \rightarrow L(W)$, defined on monomials by

$$L\varphi(x_1 \otimes \cdots \otimes x_{n-1}) = \varphi(x_1) \otimes \cdots \otimes \varphi(x_{n-1}) \quad (34)$$

and extended to all of $L(V)$ by linearity. It is clear from the definition that if $\varphi : V \rightarrow W$ and $\psi : U \rightarrow V$ are morphisms, then $L(\varphi \circ \psi) = L\varphi \circ L\psi$.

Lemma 6. *The map $L\varphi : L(V) \rightarrow L(W)$ defined in (34) is a morphism of Leibniz algebras.*

Proof. Recall that a linear map $\varphi : V \rightarrow W$ is a morphism of n -Leibniz algebras if

$$\varphi([x_1, \dots, x_n]_V) = [\varphi(x_1), \dots, \varphi(x_n)]_W. \quad (35)$$

If this is the case, then the Leibniz bracket of two monomials in the image of $L\varphi$ is given by

$$\begin{aligned} & [L\varphi(x_1 \otimes \cdots \otimes x_{n-1}), L\varphi(y_1 \otimes \cdots \otimes y_{n-1})] \\ &= [\varphi(x_1) \otimes \cdots \otimes \varphi(x_{n-1}), \varphi(y_1) \otimes \cdots \otimes \varphi(y_{n-1})] \\ &= D(\varphi(x_1) \otimes \cdots \otimes \varphi(x_{n-1})) \cdot (\varphi(y_1) \otimes \cdots \otimes \varphi(y_{n-1})) \\ &= \sum_{i=1}^{n-1} \varphi(y_1) \otimes \cdots \otimes [\varphi(x_1), \dots, \varphi(x_{n-1}), \varphi(y_i)]_W \otimes \cdots \otimes \varphi(y_{n-1}) \\ &= \sum_{i=1}^{n-1} \varphi(y_1) \otimes \cdots \otimes \varphi[x_1, \dots, x_{n-1}, y_i]_V \otimes \cdots \otimes \varphi(y_{n-1}) \\ &= \sum_{i=1}^{n-1} L\varphi(y_1 \otimes \cdots \otimes [x_1, \dots, x_{n-1}, y_i]_V \otimes \cdots \otimes y_{n-1}) \\ &= L\varphi \left(\sum_{i=1}^{n-1} y_1 \otimes \cdots \otimes [x_1, \dots, x_{n-1}, y_i]_V \otimes \cdots \otimes y_{n-1} \right) \\ &= L\varphi(D(x_1 \otimes \cdots \otimes x_{n-1}) \cdot (y_1 \otimes \cdots \otimes y_{n-1})) \\ &= L\varphi[x_1 \otimes \cdots \otimes x_{n-1}, y_1 \otimes \cdots \otimes y_{n-1}]. \end{aligned}$$

□

There are special classes of n -Leibniz algebras defined by imposing symmetry conditions on the n -bracket. The best known are the n -Lie algebras, where the bracket is alternating, but there are others. In the case of n -Lie algebras, the n -bracket defines a map $D : \Lambda^{n-1}V \rightarrow \text{End } V$ satisfying the fundamental identity (3). Just as we did above for the general n -Leibniz algebra, we may define now on $\Lambda^{n-1}V$ the structure of a Leibniz algebra in such a way that D is a Leibniz algebra morphism. The formula for the bracket in $\Lambda^{n-1}V$ is formally identical to equation (31), except that $X, Y \in \Lambda^{n-1}V$. This closes because $\Lambda^{n-1}V$ is a $\mathfrak{gl}(V)$ -submodule of $V^{\otimes(n-1)}$, whence $D(X) \cdot Y \in \Lambda^{n-1}V$. This clearly generalises to other n -Leibniz algebras where the n -bracket defines a map $D : T(V) \rightarrow \text{End } V$, for $T(V)$ a $\mathfrak{gl}(V)$ -submodule of $V^{\otimes(n-1)}$ which becomes a Leibniz algebra in its own right by virtue of equation (31).

Actually $D : T(V) \rightarrow \text{End } V$ is strictly weaker than the conditions on the bracket. We really ought to say that the bracket maps $B(V) \rightarrow V$, where $B(V) \subset V^{\otimes n}$ is a $\mathfrak{gl}(V)$ -submodule. This then induces $D : T(V) \rightarrow \text{End } V$, where $B(V) \subset T(V) \otimes V$. For example, for an n -Lie algebra, $B(V) = \Lambda^n V$ whereas $T(V) = \Lambda^{n-1}V$.

3.2. From metric n -Leibniz algebras to metric Leibniz algebras. An important class of n -Leibniz algebras, due to their appearance in a number of physical contexts, are those which possess an inner product (here, a nondegenerate symmetric bilinear form) which is invariant under inner derivations. We will show that in this case, the associated Leibniz algebra itself possesses a left-invariant inner product. Let us recall that an n -Leibniz algebra V is said to be **metric**, if V admits an inner product $\langle -, - \rangle$ which is invariant under inner derivations; that is, for all $x_1, \dots, x_{n-1}, y, z \in V$, one has

$$\langle [x_1, \dots, x_{n-1}, y], z \rangle + \langle y, [x_1, \dots, x_{n-1}, z] \rangle = 0, \quad (36)$$

or equivalently,

$$\langle D(X) \cdot y, z \rangle + \langle y, D(X) \cdot z \rangle = 0, \quad (37)$$

for all $y, z \in V$ and $X \in V^{\otimes(n-1)}$.

Proposition 7. *Let V be a metric n -Leibniz algebra and let $L(V) = V^{\otimes n}$ be its associated Leibniz algebra. Then the natural inner product on $L(V)$, defined on monomials by*

$$\langle x_1 \otimes \dots \otimes x_{n-1}, y_1 \otimes \dots \otimes y_{n-1} \rangle = \prod_{i=1}^{n-1} \langle x_i, y_i \rangle \quad (38)$$

and later extending linearly to all of $L(V)$, is invariant under left multiplication in $L(V)$; that is, for all $X, Y, Z \in L(V)$:

$$\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle = 0. \quad (39)$$

Proof. Let $X = x_1 \otimes \dots \otimes x_{n-1}$ and $Y = y_1 \otimes \dots \otimes y_{n-1}$ and let $Z \in L(V)$. Then the left-hand side of equation (39) expands to

$$\begin{aligned} \langle [Z, X], Y \rangle &= \langle [Z, x_1 \otimes \dots \otimes x_{n-1}], y_1 \otimes \dots \otimes y_{n-1} \rangle \\ &= \langle D(Z) \cdot (x_1 \otimes \dots \otimes x_{n-1}), y_1 \otimes \dots \otimes y_{n-1} \rangle \\ &= \sum_{i=1}^{n-1} \langle x_1 \otimes \dots \otimes D(Z)x_i \otimes \dots \otimes x_{n-1}, y_1 \otimes \dots \otimes y_{n-1} \rangle \\ &= \sum_{i=1}^{n-1} \langle D(Z)x_i, y_i \rangle \prod_{\substack{j=1 \\ j \neq i}}^{n-1} \langle x_j, y_j \rangle && \text{by equation (38)} \\ &= - \sum_{i=1}^{n-1} \langle x_i, D(Z)y_i \rangle \prod_{\substack{j=1 \\ j \neq i}}^{n-1} \langle x_j, y_j \rangle && \text{by equation (37)} \\ &= - \sum_{i=1}^{n-1} \langle x_1 \otimes \dots \otimes x_{n-1}, y_1 \otimes \dots \otimes D(Z)y_i \otimes \dots \otimes y_{n-1} \rangle \\ &= - \langle x_1 \otimes \dots \otimes x_{n-1}, D(Z) \cdot (y_1 \otimes \dots \otimes y_{n-1}) \rangle \\ &= - \langle X, [Z, Y] \rangle. \end{aligned}$$

□

The assignment $V \mapsto L(V)$ is still functorial between the categories of metric n -Leibniz algebras and metric Leibniz algebras. Indeed, we have the following

Proposition 8. *Let $\varphi : V \rightarrow W$ is a morphism of metric n -Leibniz algebras, so that in addition to equation (35), it is also an isometry:*

$$\langle \varphi(x), \varphi(y) \rangle_W = \langle x, y \rangle_V . \quad (40)$$

Then $L\varphi : L(V) \rightarrow L(W)$ defined by equation (34) is a morphism of metric Leibniz algebras.

Proof. Lemma 6 says that $L\varphi$ preserves the bracket, whence it remains to show that it is an isometry. Let $X, Y \in L(V)$ be given by $x_1 \otimes \cdots \otimes x_{n-1}$ and $y_1 \otimes \cdots \otimes y_{n-1}$, respectively. Then

$$\begin{aligned} \langle L\varphi(X), L\varphi(Y) \rangle &= \langle \varphi(x_1) \otimes \varphi(x_{n-1}), \varphi(y_1) \otimes \varphi(y_{n-1}) \rangle \\ &= \prod_{i=1}^{n-1} \langle \varphi(x_i), \varphi(y_i) \rangle_W && \text{by equation (38)} \\ &= \prod_{i=1}^{n-1} \langle x_i, y_i \rangle_V && \text{by equation (40)} \\ &= \langle x_1 \otimes \cdots \otimes x_{n-1}, y_1 \otimes \cdots \otimes y_{n-1} \rangle && \text{by equation (38)} \\ &= \langle X, Y \rangle . \end{aligned}$$

□

The above two propositions remain true for metric n -Lie algebras, where now $L(V) = \Lambda^{n-1}V$ and the induced inner product takes the standard determinantal form

$$\langle x_1 \wedge \cdots \wedge x_{n-1}, y_1 \wedge \cdots \wedge y_{n-1} \rangle = \det(\langle x_i, y_j \rangle) , \quad (41)$$

since the essential point is that the inner product on V is invariant under the action of $D(X)$ for any $X \in L(V)$. One final remark is that, although as shown in [4] the image of D in $\mathfrak{so}(V)$ is itself a metric Lie algebra, D is not in general an isometry. This is easily illustrated by the unique simple euclidean 3-Lie algebra, discussed from the Faulkner point of view in [4, Example 4] and from the Leibniz point of view below as Example 28. Let us simply point out that here $V = \mathbb{R}^4$ and $L(V) = \Lambda^2\mathbb{R}^4$ and the Lie algebra of inner derivations is $\mathfrak{so}(4)$. The map $D : \Lambda^2\mathbb{R}^4 \rightarrow \mathfrak{so}(4)$ is in this case an isomorphism of Leibniz (and hence of Lie) algebras, but it is not an isometry because whereas the natural inner product on $L(V)$ has positive signature, the one on $\mathfrak{so}(4)$ has split signature.

4. DEFORMATIONS OF AN n -LEIBNIZ ALGEBRA

In this section we reinterpret the deformation theory of n -Leibniz algebras in terms of the cohomology of its associated Leibniz algebra.

4.1. Deformation complex. Let V be an n -Leibniz algebra with associated Leibniz algebra $L = L(V)$. Both the algebraic structures on V and on $L(V)$ are given by the Leibniz algebra morphism $D : L \rightarrow \mathfrak{gl}(V)$. By a **deformation** of V (in the sense of Gerstenhaber) we mean an analytic one-parameter family of n -Leibniz algebras on V defined by a bracket

$$[x_1, \dots, x_n]_t = [x_1, \dots, x_n] + \sum_{k \geq 1} t^k \Phi_k(x_1, \dots, x_n) , \quad (42)$$

where $\Phi_k : V^n \rightarrow V$ are multilinear maps. Such a bracket gives rise to a family of maps D_t defined by

$$D_t = D + \sum_{k \geq 1} t^k \varphi_k , \quad (43)$$

where $\varphi_k : L \rightarrow \text{End } V$ is defined by

$$\varphi_k(x_1 \otimes \cdots \otimes x_{n-1}) \cdot y = \Phi_k(x_1, \dots, x_{n-1}, y) , \quad (44)$$

for all $y \in V$.

Differentiating the fundamental identity (29) for D_t at $t = 0$, we obtain the following condition on $\varphi := \varphi_1 \in CL^1(L; \text{End } V)$:

$$[D(X), \varphi(Y)] + [\varphi(X), D(Y)] - D(\varphi(X) \cdot Y) - \varphi(D(X) \cdot Y) = 0 . \quad (45)$$

Comparing this with equation (27), but with φ replacing θ , we see that this can be written as $d\varphi = 0$, provided that we define the actions of L on $\text{End } V$ as

$$[X, \psi] = [D(X), \psi] \quad \text{and} \quad [\psi, X] = [\psi, D(X)] - D(\psi \cdot X) , \quad (46)$$

for all $\psi \in \text{End } V$ and where the brackets on the right-hand sides are commutators on $\text{End } V$.

Proposition 9. *With respect to the above actions, $\text{End } V$ is a representation of $L(V)$.*

Proof. We need to show that the actions in (46) satisfy the three compatibility conditions (13), (14) and (15), making $\text{End } V$ into a representation of L . Indeed, equation (13) is clear:

$$\begin{aligned} [[X, Y], \psi] - [X, [Y, \psi]] + [Y, [X, \psi]] &= [D([X, Y]), \psi] - [D(X), [D(Y), \psi]] + [D(Y), [D(X), \psi]] \\ &= [D([X, Y]), \psi] - [[D(X), D(Y)], \psi] = 0 , \end{aligned}$$

by virtue of the fundamental identity (3). To check equations (14) and (15), it is enough to check one of them and equation (16). Let us check this latter equation:

$$\begin{aligned} [[X, \psi], Y] + [[\psi, X], Y] &= -[D(\psi \cdot X), Y] \\ &= -[D(\psi \cdot X), D(Y)] + D(D(\psi \cdot X) \cdot Y) = 0 , \end{aligned}$$

again by virtue of the fundamental identity (3) but applied to $\psi \cdot X$ and Y . Finally, we check equation (14). Using the fundamental identity, the left-hand side expands to

$$\begin{aligned} [[X, \psi], Y] &= [[D(X), \psi], Y] \\ &= [[D(X), \psi], D(Y)] - D([D(X), \psi] \cdot Y) \\ &= [D(X), [\psi, D(Y)]] - [\psi, [D(X), D(Y)]] - D(D(X) \cdot \psi \cdot Y) + D(\psi \cdot D(X) \cdot Y) \\ &= [D(X), [\psi, D(Y)]] - [\psi, D([X, Y])] - [D(X), D(\psi \cdot Y)] + D(\psi \cdot [X, Y]) , \end{aligned}$$

whereas the right-hand side expands to the same thing:

$$\begin{aligned} [X, [\psi, Y]] - [\psi, [X, Y]] &= [D(X), [\psi, Y]] - [\psi, D([X, Y])] + D(\psi \cdot [X, Y]) \\ &= [D(X), [\psi, D(Y)]] - [D(X), D(\psi \cdot Y)] \\ &\quad - [\psi, D([X, Y])] + D(\psi \cdot [X, Y]) . \end{aligned}$$

□

Notice that this representation is *not* symmetric, whence it is not induced from a representation of the associated Lie algebra \mathfrak{g}_L .

A deformation is said to be **trivial** if it is due to the action of a one-parameter subgroup g_t of the general linear group $\text{GL}(V)$; that is, if

$$g_t([x_1, \dots, x_n]_t) = [g_t(x_1), \dots, g_t(x_n)] , \quad (47)$$

or equivalently

$$g_t \circ D_t(X) = D(g_t \cdot X) \circ g_t , \quad (48)$$

for all $X \in L$. Let $g_t(x) = x + t\gamma(x) + O(t^2)$ and differentiate the above equation at $t = 0$ to obtain

$$\varphi(X) = -[\gamma, D(X)] + D(\gamma \cdot X) = -[\gamma, X] , \quad (49)$$

whence $\varphi = d\gamma$.

In other words, we have proved the following

Theorem 10. *Isomorphism classes of infinitesimal deformations of the n -Leibniz algebra V are classified by $HL^1(L(V); \text{End } V)$, with $\text{End } V$ the nonsymmetric representation of $L(V)$ defined by equation (46).*

If all deformations of L are trivial, we say that the n -Leibniz algebra L is **rigid**. A sufficient condition for rigidity is the vanishing of $HL^1(L(V); \text{End } V)$, but this is not necessary, since infinitesimal deformations might be obstructed, as we now review.

4.2. Obstructions. Given an infinitesimal deformation of an n -Leibniz algebra, one would like to know whether it integrates to a one-parameter deformation. Based on one's experience with the deformation theory of other algebraic structures, one expects an infinite sequence of obstructions (each one defined provided the previous one is overcome) living in the same cohomology theory as the infinitesimal deformations but one dimension higher. Furthermore, one expects these obstruction classes to be given by universal formulae using a natural graded Lie algebra structure on the cohomology, as explained for Lie algebras by Nijenhuis and Richardson in [16]. We will see that this is indeed the case in the next section, but for now let us illustrate this by trying to integrate an infinitesimal deformation to second order.

We write the deformed n -bracket on V as

$$[x_1, \dots, x_n]_t = [x_1, \dots, x_n] + t\Phi_1(x_1, \dots, x_n) + t^2\Phi_2(x_1, \dots, x_n) + O(t^3) , \quad (50)$$

giving rise to $D_t : V^{\otimes(n-1)} \rightarrow \text{End } V$ defined by

$$D_t(X) = D(X) + t\varphi_1(X) + t^2\varphi_2(X) + O(t^3) , \quad (51)$$

where $\varphi_i \in CL^1(L; \text{End } V)$. Expanding the fundamental identity (3) for D_t to order t^2 , one finds to zeroth order the fundamental identity for D , to first order the cocycle condition for φ_1 and to second order the following identity

$$\begin{aligned} [D(X), \varphi_2(Y)] + [\varphi_2(X), D(Y)] + [\varphi_1(X), \varphi_1(Y)] \\ = D(\varphi_2(X) \cdot Y) + \varphi_2(D(X) \cdot Y) + \varphi_1(\varphi_1(X) \cdot Y) , \end{aligned}$$

which we recognise as

$$d\varphi_2(X, Y) = \varphi_1(\varphi_1(X) \cdot Y) - [\varphi_1(X), \varphi_1(Y)] . \quad (52)$$

It is a straightforward calculation, using that φ_1 is a cocycle, to show that the right-hand side of this equation defines a cocycle in $CL^2(L; \text{End } V)$ whose cohomology class is the obstruction to integrability (to second order), since if and only if this class vanishes, can we find φ_2 obeying equation (52). We will be able to interpret the right-hand side of (52) as a bracket $-\frac{1}{2}[\varphi_1, \varphi_1]$ in $HL^2(L; \text{End } V)$ analogous to the Nijenhuis–Richardson [16] bracket on the Chevalley–Eilenberg cohomology $H^\bullet(\mathfrak{g}; \mathfrak{g})$ of a Lie algebra \mathfrak{g} . This will allow us to prove in complete generality that the obstructions to integrating an infinitesimal deformation are cohomology classes in $HL^2(L; \text{End } V)$.

4.3. Another look at the deformation complex. We may understand the deformation complex in a slightly different way, which serves to illustrate a number of things. First of all, we notice that a deformation of the n -Leibniz algebra V implies a deformation of the underlying Leibniz algebra $L(V)$. However the notions of trivial deformations do not agree. A deformation of the Leibniz algebra $L(V)$ is trivial if it is due to the action of a one-parameter subgroup of $\mathrm{GL}(L(V))$, but since not every invertible linear transformation of $L(V)$ is induced from one of V , we may have that a deformation of $L(V)$ may be trivial without the corresponding deformation of V being trivial.

A deformation of the Leibniz algebra $L := L(V)$ takes the form

$$[X, Y]_t = [X, Y] + t\Psi(X, Y) + O(t^2) , \quad (53)$$

where $\Psi : L^2 \rightarrow L$ is a bilinear map. Expanding the Leibniz identity (8) for the deformed bracket to first order recovers, at zeroth order, the Leibniz identity for the undeformed bracket and, at first order, the following equation for Ψ :

$$\begin{aligned} [X, \Psi(Y, Z)] - [Y, \Psi(X, Z)] - [\Psi(X, Y), Z] \\ + \Psi(X, [Y, Z]) - \Psi(Y, [X, Z]) - \Psi([X, Y], Z) = 0 . \end{aligned} \quad (54)$$

Comparing with the expression (28) for the differential in Leibniz cohomology, we see that this is the cocycle condition for $\Psi \in CL^2(L; L)$. The deformation is trivial if it is the result of the action of a one-parameter subgroup g_t of the general linear group $\mathrm{GL}(L)$, so that

$$g_t([X, Y]_t) = [g_t(X), g_t(Y)] . \quad (55)$$

Letting $g_t(X) = X + t\gamma(X) + O(t^2)$ and differentiating the above equation with respect to t at $t = 0$ we obtain

$$\Psi(X, Y) = [X, \gamma(Y)] + [\gamma(X), Y] - \gamma([X, Y]) , \quad (56)$$

whence $\Psi = d\gamma$, for $\gamma \in CL^1(L; L)$.

This proves the following

Theorem 11. *Infinitesimal deformations of a Leibniz algebra L are classified by $HL^2(L; L)$.*

Now we have a vector space isomorphism

$$CL^{p+1}(L; L) = \mathrm{Hom}(L^{\otimes(p+1)}, L) \cong \mathrm{Hom}(L^{\otimes p}, \mathrm{End} L) = CL^p(L; \mathrm{End} L) , \quad (57)$$

for $p \geq 0$. We may promote this to an isomorphism of complexes by defining the differential on $CL^\bullet(L; \mathrm{End} L)$ appropriately, which will tell us in turn how to view $\mathrm{End} L$ as a representation of L .

In lowest dimension, we must impose the commutativity of the following diagram:

$$\begin{array}{ccc} CL^1(L; L) & \xrightarrow{\cong} & CL^0(L; \mathrm{End} L) \\ d \downarrow & & \downarrow d \\ CL^2(L; L) & \xrightarrow{\cong} & CL^1(L; \mathrm{End} L) \\ d \downarrow & & \downarrow d \\ CL^3(L; L) & \xrightarrow{\cong} & CL^2(L; \mathrm{End} L) , \end{array} \quad (58)$$

for some suitable $d : CL^p(L; \mathrm{End} L) \rightarrow CL^{p+1}(L; \mathrm{End} L)$ determined by how L acts on $\mathrm{End} L$. It is this action which we will determine.

Consider $\psi \in CL^1(L; L) = \mathrm{End} L$. Then for $\psi \in CL^1(L; L)$, $d\psi \in CL^2(L; L)$ is given by

$$d\psi(X, Y) = [X, \psi(Y)] + [\psi(X), Y] - \psi([X, Y]) . \quad (59)$$

On the other hand, for $\psi \in CL^0(L; \text{End } L) = \text{End } L$,

$$d\psi(X) = -[\psi, X] , \quad (60)$$

whence demanding commutativity of the top square,

$$d\psi(X)(Y) = -[\psi, X](Y) = [X, \psi(Y)] + [\psi(X), Y] - \psi([X, Y]) , \quad (61)$$

which says that the right action of L on $\text{End } L$ is given by

$$[\psi, X](Y) = -[X, \psi(Y)] - [\psi(X), Y] + \psi([X, Y]) . \quad (62)$$

Now let $\Phi \in CL^2(L; L)$ and let the corresponding element in $CL^1(L; \text{End } L)$ be φ ; that is, $\varphi(X)(Y) = \Phi(X, Y)$. Then on the one hand,

$$\begin{aligned} d\Phi(X, Y, Z) &= [X, \Phi(Y, Z)] - [Y, \Phi(X, Z)] - [\Phi(X, Y), Z] \\ &\quad + \Phi(X, [Y, Z]) - \Phi(Y, [X, Z]) - \Phi([X, Y], Z) , \end{aligned}$$

which we would like to equate with

$$d\varphi(X, Y) = [X, \varphi(Y)] + [\varphi(X), Y] - \varphi([X, Y]) \quad (63)$$

applied to Z :

$$d\varphi(X, Y)(Z) = [X, \varphi(Y)](Z) - [Y, \Phi(X, Z)] - [\Phi(X, Y), Z] + \Phi(X, [Y, Z]) - \Phi([X, Y], Z) ,$$

where we have used equation (62). Comparing the two expressions determines the left action of L on $\text{End } L$ to be

$$[X, \psi](Y) = [X, \psi(Y)] - \psi([X, Y]) . \quad (64)$$

Proposition 12. *With the actions defined by (62) and (64), $\text{End } L$ is a representation of L .*

Proof. We need to check that conditions (13), (14) and (15) are satisfied.

Checking condition (13) we apply it to $Z \in L$ and use the Leibniz identity for L to expand the left-hand side as follows:

$$\begin{aligned} [[X, Y], \psi](Z) &= [[X, Y], \psi(Z)] - \psi([[X, Y], Z]) \\ &= [X, [Y, \psi(Z)]] - [Y, [X, \psi(Z)]] - \psi([X, [Y, Z]]) + \psi([Y, [X, Z]]) , \end{aligned}$$

whereas expanding the right-hand side we obtain, for the first term

$$\begin{aligned} [X, [Y, \psi]](Z) &= [X, [Y, \psi](Z)] - [X, \psi]([Y, Z]) \\ &= [X, [Y, \psi(Z)]] - [X, \psi([Y, Z])] - [Y, \psi([X, Z])] + \psi([Y, [X, Z]]) , \end{aligned}$$

and for the second term

$$\begin{aligned} -[Y, [X, \psi]](Z) &= -[Y, [X, \psi](Z)] + [Y, \psi]([X, Z]) \\ &= -[Y, [X, \psi(Z)]] + [Y, \psi([X, Z])] + [X, \psi([Y, Z])] - \psi([X, [Y, Z]]) . \end{aligned}$$

Adding them we find that four of the terms cancel pairwise and the remaining four are precisely what we obtained for the left-hand side.

In order to check equations (14) and (15), it is enough to check one of them and equation (16). Applying this latter equation to $Z \in L$ and expanding, we obtain for the first term,

$$\begin{aligned} [[X, \psi], Y](Z) &= -[Y, [X, \psi](Z)] - [[X, \psi](Y), Z] + [X, \psi]([Y, Z]) \\ &= -[Y, [X, \psi(Z)]] + [Y, \psi([X, Z])] - [[X, \psi(Y)], Z] \\ &\quad + [\psi([X, Y]), Z] + [X, \psi([Y, Z])] - \psi([X, [Y, Z]]) , \end{aligned}$$

and for the second

$$\begin{aligned}
[[\psi, X], Y](Z) &= -[[Y, [\psi, X](Z)] - [[\psi, X](Y), Z] + [\psi, X]([Y, Z])] \\
&= [Y, [X, \psi(Z)]] + [Y, [\psi(X), Z]] - [Y, \psi([X, Z])] \\
&\quad + [[X, \psi(Y)], Z] + [[\psi(X), Y], Z] - [\psi([X, Y]), Z] \\
&\quad - [X, \psi([Y, Z])] - [\psi(X), [Y, Z]] + \psi([X, [Y, Z]]) .
\end{aligned}$$

Adding the two, six terms cancel pairwise, leaving

$$[[X, \psi], Y](Z) + [[\psi, X], Y](Z) = [Y, [\psi(X), Z]] + [[\psi(X), Y], Z] - [\psi(X), [Y, Z]] , \quad (65)$$

which vanishes because of the Leibniz identity (7) with $\psi(X)$ replacing X . Finally, we check equation (14), by applying it to Z and expanding. Doing so with the left-hand side we find

$$\begin{aligned}
[[X, \psi], Y](Z) &= -[Y, [X, \psi](Z)] - [[X, \psi](Y), Z] + [X, \psi]([Y, Z]) \\
&= -[Y, [X, \psi(Z)]] + [Y, \psi([X, Z])] - [[X, \psi(Y)], Z] \\
&\quad + [\psi([X, Y]), Z] + [X, \psi([Y, Z])] - \psi([X, [Y, Z]]) ,
\end{aligned}$$

whereas for the right-hand side we find

$$\begin{aligned}
[X, [\psi, Y]](Z) - [\psi, [X, Y]](Z) &= [X, [\psi, Y](Z)] - [\psi, Y]([X, Z]) + [[X, Y], \psi(Z)] \\
&\quad + [\psi([X, Y]), Z] - \psi([X, Y], Z]) \\
&= -[X, [Y, \psi(Z)]] - [X, [\psi(Y), Z]] + [X, \psi([Y, Z])] \\
&\quad + [Y, \psi([X, Z])] + [\psi(Y), [X, Z]] - \psi([Y, [X, Z]]) \\
&\quad + [[X, Y], \psi(Z)] + [\psi([X, Y]), Z] - \psi([X, Y], Z]) .
\end{aligned}$$

Comparing the two we find that eight terms cancel pairwise in their difference and the rest are

$$\begin{aligned}
&[Y, [X, \psi(Z)]] - [X, [Y, \psi(Z)]] + [[X, Y], \psi(Z)] \\
&\quad + [[X, \psi(Y)], Z] - [X, [\psi(Y), Z]] + [\psi(Y), [X, Z]] \\
&\quad + \psi([X, [Y, Z]]) - \psi([Y, [X, Z]]) - \psi([X, Y], Z]) ,
\end{aligned}$$

each line of which vanishes because of the Leibniz identity. \square

We therefore have two complexes $CL^{\bullet+1}(L; L)$ and $CL^{\bullet}(L; \text{End } L)$ which are isomorphic as graded vector spaces and, as seen above, also isomorphic as complexes in the lowest two degrees. In fact, we have more.

Proposition 13. *The vector space isomorphism $CL^p(L; \text{End } L) \rightarrow CL^{p+1}(L; L)$, sending $\varphi \mapsto \Phi$, where*

$$\varphi(X_1, \dots, X_p)(Y) = \Phi(X_1, \dots, X_p, Y) , \quad (66)$$

is an isomorphism of complexes.

Proof. We need to show that the map defined by (66) is a chain map. By equation (24), applying $(d\varphi)(X_1, \dots, X_{p+1})$ to $X_{p+2} \in L$, we obtain

$$\begin{aligned}
(d\varphi)(X_1, \dots, X_{p+1})(X_{p+2}) &= \sum_{i=1}^p (-1)^{i-1} [X_i, \varphi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1})](X_{p+2}) \\
&\quad + (-1)^{p+1} [\varphi(X_1, \dots, X_p), X_{p+1}](X_{p+2}) \\
&\quad + \sum_{1 \leq i < j \leq p+1} (-1)^i \varphi(X_1, \dots, \widehat{X_i}, \dots, [X_i, X_j], \dots, X_{p+1})(X_{p+2}) ,
\end{aligned}$$

which becomes, using equations (62) and (64),

$$\begin{aligned}
&= \sum_{i=1}^p (-1)^{i-1} [X_i, \Phi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1}, X_{p+2})] \\
&\quad - \sum_{i=1}^p (-1)^{i-1} \Phi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1}, [X_i, X_{p+2}]) \\
&\quad - (-1)^{p+1} [X_{p+1}, \Phi(X_1, \dots, X_p, X_{p+2})] \\
&\quad - (-1)^{p+1} [\Phi(X_1, \dots, X_{p+1}), X_{p+2}] \\
&\quad + (-1)^{p+1} \Phi(X_1, \dots, X_p, [X_{p+1}, X_{p+2}]) \\
&\quad + \sum_{1 \leq i < j \leq p+1} (-1)^i \Phi(X_1, \dots, \widehat{X_i}, \dots, [X_i, X_j], \dots, X_{p+1}, X_{p+2}) .
\end{aligned}$$

The first and third terms make up

$$\sum_{i=1}^{p+1} (-1)^{i-1} [X_i, \Phi(X_1, \dots, \widehat{X_i}, \dots, X_{p+2})] , \tag{67}$$

whereas the second term and the last two make up

$$\sum_{1 \leq i < j \leq p+2} (-1)^i \Phi(X_1, \dots, \widehat{X_i}, \dots, [X_i, X_j], \dots, X_{p+2}) . \tag{68}$$

Putting everything together we arrive at

$$\begin{aligned}
(d\varphi)(X_1, \dots, X_{p+1})(X_{p+2}) &= \sum_{i=1}^{p+1} (-1)^{i-1} [X_i, \Phi(X_1, \dots, \widehat{X_i}, \dots, X_{p+2})] \\
&\quad + (-1)^{p+2} [\Phi(X_1, \dots, X_{p+1}), X_{p+2}] \\
&\quad + \sum_{1 \leq i < j \leq p+2} (-1)^i \Phi(X_1, \dots, \widehat{X_i}, \dots, [X_i, X_j], \dots, X_{p+2}) \\
&= d\Phi(X_1, \dots, X_{p+2}) ,
\end{aligned}$$

using equation (24), whence the isomorphism (66) is a chain map and hence an isomorphism of complexes. \square

Together with Theorem 11, the above isomorphism of complexes implies the following

Theorem 14. *Infinitesimal deformations of a Leibniz algebra L are classified by $HL^1(L; \text{End } L)$ with $\text{End } L$ the representation of L defined by (62) and (64).*

Back to the deformations of an n -Leibniz algebra V , since $L(V) = V^{\otimes(n-1)}$ is a faithful $\text{GL}(V)$ -module, we have an injective map $\iota : \text{End } V \rightarrow \text{End } L(V)$. If $\psi \in \text{End } V$, its image $\iota(\psi) \in \text{End } L(V)$ is such that if $x_1 \otimes \dots \otimes x_{n-1} \in L(V)$ is a monomial,

$$\iota(\psi)(x_1 \otimes \dots \otimes x_{n-1}) = \sum_{i=1}^{n-1} x_1 \otimes \dots \otimes \psi(x_i) \otimes \dots \otimes x_{n-1} , \tag{69}$$

and we extend to all of $L(V)$ by linearity. We will often use the shorthand notation $\psi \cdot X$ for $\iota(\psi)(X)$, as was done in (5), for instance.

Proposition 15. *The map $\iota : \text{End } V \rightarrow \text{End } L(V)$ is a morphism of $L(V)$ representations, where $L(V)$ acts on $\text{End } V$ according to (46).*

Proof. We want to show that, for all $X \in L(V)$ and $\psi \in \text{End } V$, the following relations hold:

$$[X, \iota(\psi)] = \iota([X, \psi]) \quad \text{and} \quad [\iota(\psi), X] = \iota([\psi, X]) . \quad (70)$$

For $Y \in L(V)$ we have

$$\begin{aligned} [X, \iota(\psi)](Y) &= [X, \psi \cdot Y] - \psi \cdot [X, Y] && \text{by equation (64)} \\ &= D(X) \cdot \psi \cdot Y - \psi \cdot D(X) \cdot Y \\ &= [D(X), \psi] \cdot Y \\ &= \iota([X, \psi])(Y) && \text{by the first equation in (46).} \end{aligned}$$

Similarly,

$$\begin{aligned} [\iota(\psi), X](Y) &= -[X, \iota(\psi)(Y)] - [\iota(\psi)(X), Y] + \iota(\psi)([X, Y]) && \text{by equation (62)} \\ &= -D(X) \cdot \psi \cdot Y - D(\psi \cdot X) \cdot Y + \psi \cdot D(X) \cdot Y \\ &= [\psi, D(X)] \cdot Y - D(\psi \cdot X) \cdot Y \\ &= [\psi, X] \cdot Y && \text{by the second equation in (46)} \\ &= \iota([\psi, X])(Y) . \end{aligned}$$

□

This map induces an injective map of complexes $CL^\bullet(L; \text{End } V) \rightarrow CL^\bullet(L; \text{End } L)$, whence the deformation complex for the n -Leibniz algebra written in Section 4.1 is a subcomplex of the deformation complex of the associated Leibniz algebra $L(V)$. It is preferable, however, to work with $CL^\bullet(L; \text{End } V)$ itself.

4.4. A graded Lie algebra structure on the deformation complex. In the study of deformations of Lie algebras, many calculations become simpler by first exhibiting a graded Lie algebra structure on the deformation complex, relative to which the differential is an inner derivation. As a consequence, the cocycles are a (graded Lie) subalgebra of which the coboundaries are an ideal, whence the cohomology itself inherits the structure of a graded Lie algebra. The same situation obtains in the deformation theory of n -Leibniz algebras.

Let us depart from the observation that if V is an n -Leibniz algebra with associated Leibniz algebra L , then $D : L \rightarrow \text{End } V$ can be understood as a cochain $D \in CL^1(L; \text{End } V)$. This cochain is actually a cocycle:

$$\begin{aligned} dD(X, Y) &= [X, D(Y)] + [D(X), Y] - D([X, Y]) \\ &= [D(X), D(Y)] + [D(X), D(Y)] - D(D(X) \cdot Y) - D([X, Y]) \\ &= 2([D(X), D(Y)] - D([X, Y])) = 0 , \end{aligned}$$

by the fundamental identity. Furthermore D is a coboundary. Indeed, let $\mathbf{1} \in \text{End } V$ denote the identity endomorphism and consider

$$d\mathbf{1}(X) = -[\mathbf{1}, X] = -[\mathbf{1}, D(X)] + D(\mathbf{1} \cdot X) = (n-1)D(X) , \quad (71)$$

whence $D = \frac{1}{n-1}d\mathbf{1}$.

If we define, for $\alpha, \beta \in CL^1(L; \text{End } V)$, their bracket $[\alpha, \beta] \in CL^2(L; \text{End } V)$ by

$$[\alpha, \beta](X, Y) = [\alpha(X), \beta(Y)] - \alpha(\beta(X) \cdot Y) + [\beta(X), \alpha(Y)] - \beta(\alpha(X) \cdot Y) , \quad (72)$$

then we have that $[D, \beta] = d\beta$ and that $[D, D] = 0$ because of the fundamental identity, whence $d^2 = 0$. This suggests very strongly the following: the bracket above extends to a graded Lie bracket on all of $CL^\bullet(L; \text{End } V)$ and the differential in the complex is given by $[D, -]$. This

turns out to be the case and the existence of this graded Lie algebra structure on $CL^\bullet(L; \text{End } V)$ can be deduced in at least two ways:

- (1) from work of Balavoine [17] for Leibniz algebras, via the maps

$$CL^\bullet(L; \text{End } V) \hookrightarrow CL^\bullet(L; \text{End } L) \cong CL^{\bullet+1}(L; L) . \quad (73)$$

The latter map pulls back to $CL^\bullet(L; \text{End } L)$ the graded Lie algebra structure on $CL^{\bullet+1}(L; L)$ defined by Balavoine, and one checks that $CL^\bullet(L; \text{End } V)$ is a graded Lie subalgebra; or

- (2) more directly from work of Rotkiewicz [18] who defines a graded Lie algebra structure for a cohomology complex associated to an n -Leibniz algebra, which is isomorphic to $CL^\bullet(L; \text{End } V)$.

In either case, we have the following

Theorem 16 (Balavoine [17], Rotkiewicz [18]). *The complex $CL^\bullet(L; \text{End } V)$ admits the structure of a graded Lie algebra in such a way that the differential is an inner derivation $d = [D, -]$ by an element $D \in CL^1(L; \text{End } V)$ obeying $[D, D] = 0$.*

In what follows we will not need the explicit expression for the graded Lie bracket; it will be enough to know that it exists and that in the expression $[\alpha, \beta](X_1, \dots, X_{p+q})$ where $\alpha \in CL^p(L; \text{End } V)$ and $\beta \in CL^q(L; \text{End } V)$ and $X_i \in L$, there are only two kinds of terms: commutators of the form $[\alpha(X_{i_1}, \dots, X_{i_p}), \beta(X_{i_{p+1}}, \dots, X_{i_{p+q}})]$ in $\text{End } V$ and terms in the image of either α or β as in the example in equation (72).

Since the fundamental identity is equivalent to $[D, D] = 0$, we may consider the fundamental identity of the deformation $[D_t, D_t] = 0$ and expand it in powers of t . Let us assume that we have a deformation to order t^N . This means that we have

$$D_N = D + \sum_{k=1}^N t^k \varphi_k \quad (74)$$

satisfying

$$[D_N, D_N] = t^{N+1} \xi + O(t^{N+2}) . \quad (75)$$

We claim that ξ is a cocycle. This is equivalent to showing that $[D, [D_N, D_N]] = O(t^{N+2})$, but this is clear because

$$[D, [D_N, D_N]] = [D_N - (D_N - D), [D_N, D_N]] = [D_N, [D_N, D_N]] - [D_N - D, [D_N, D_N]] , \quad (76)$$

and the first term in the right-hand side vanishes because of the Jacobi identity and, since $D_N - D = O(t)$, the second term is $O(t^{N+2})$ as desired.

Furthermore if and only if the class of ξ in $HL^2(L; \text{End } V)$ vanishes, so that $\xi = -2d\varphi_{N+1}$, can we extend the deformation to the next order by defining

$$D_{N+1} := D_N + t^{N+1} \varphi_{N+1} \quad (77)$$

and noticing that now $[D_{N+1}, D_{N+1}] = O(t^{N+2})$. In this way we arrive at an infinite sequence of obstructions in $HL^2(L; \text{End } V)$ for integrating an infinitesimal deformation.

In summary, we have the following more complete version of Theorem 10.

Theorem 17. *Infinitesimal deformations of an n -Leibniz algebra V are classified by $HL^1(L; \text{End } V)$ with $\text{End } V$ the representation of $L = V^{\otimes(n-1)}$ defined by (46). The obstructions to integrating an infinitesimal deformation live in $HL^2(L; \text{End } V)$.*

It follows that if $HL^1(L; \text{End } V) = 0$, the n -Leibniz algebra L is rigid, whereas infinitesimal deformations are unobstructed if $HL^2(L; \text{End } V) = 0$. Of course, even if $HL^1(L; \text{End } V) \neq 0$, an infinitesimal deformation may fail to integrate and L may still be rigid. Similarly, even if $HL^2(L; \text{End } V) \neq 0$, infinitesimal deformations may still be unobstructed. We end with the remark that because $\text{End } V$ is not a symmetric representation, one cannot use Lie algebra cohomology to compute $HL^\bullet(L; \text{End } V)$, whence it seems that these groups have to be computed using brute force.

4.5. Deformations of n -Lie algebras. Now we consider the case of n -Lie algebras, where the bracket is totally skewsymmetric and hence the associated Leibniz algebra is $L = \Lambda^{n-1}V$. It is clear that the preceding discussion applies *mutatis mutandis* (which here simply means replacing $\Lambda^{n-1}V$ for $V^{\otimes(n-1)}$ everywhere), except for one important difference: not every cocycle in $CL^1(L; \text{End } V)$ gives rise to deformation of the n -Lie algebra: we still have to impose that the resulting n -bracket be totally skewsymmetric, for whereas every totally skewsymmetric n -bracket defines a map $\Lambda^{n-1}V \rightarrow \text{End } V$, the converse does not hold: a map $\Lambda^{n-1}V \rightarrow \text{End } V$ defines an n -bracket $\Lambda^{n-1}V \otimes V \rightarrow V$, which may or may not be skewsymmetric. We may circumvent this problem by defining a subcomplex C^\bullet of $CL^\bullet(L; \text{End } V)$ which agrees with $CL^\bullet(L; \text{End } V)$ for $p \neq 1$ and such that $C^1 \subsetneq CL^1(L; \text{End } V)$ consists of those $\varphi : L \rightarrow \text{End } V$ such that the associated n -linear map

$$\Phi(x_1, \dots, x_n) := \varphi(x_1, \dots, x_{n-1})(x_n) \quad (78)$$

is totally skewsymmetric.

Lemma 18. *The subspace C^\bullet so defined is a subcomplex of $CL^\bullet(L; \text{End } V)$.*

Proof. We need only verify that the image of the differential $d : CL^0(L; \text{End } V) \rightarrow CL^1(L; \text{End } V)$ actually lives inside C^1 . To this end let $f \in \text{End } V = CL^0(L; \text{End } V)$. Let $X = x_1 \wedge \dots \wedge x_{n-1} \in L$ and consider the n -linear map associated with $df \in CL^1(L; \text{End } V)$:

$$\begin{aligned} df(X)(x_n) &= -[f, X](x_n) \\ &= -[f, D(X)](x_n) + D(f \cdot X)(x_n) \\ &= -f(D(X) \cdot x_n) + D(X) \cdot f(x_n) + D(f \cdot X)(x_n) \\ &= -f([x_1, \dots, x_n]) + [x_1, \dots, x_{n-1}, f(x_n)] + \sum_{i=1}^{n-1} [x_1, \dots, f(x_i), \dots, x_n] \\ &= -f([x_1, \dots, x_n]) + \sum_{i=1}^n [x_1, \dots, f(x_i), \dots, x_n], \end{aligned}$$

which is clearly skewsymmetric in the x_i . □

As a corollary of the proof of the previous lemma, we see that if the n -bracket is such that it maps $B(V) \rightarrow V$, for some $\mathfrak{gl}(V)$ -submodule $B(V) \subset V^{\otimes n}$, then $df \in CL^1(L; \text{End } V)$ will be such that its associated n -linear map also maps $B(V) \rightarrow V$. This follows because the calculation in the proof shows that

$$df(X)(x_n) = \Phi(f \cdot (X \otimes x_n)) - f(\Phi(X \otimes x_n)) = -(f \cdot \Phi)(X \otimes x_n), \quad (79)$$

where Φ stands for the n -bracket in V and where $f \in \text{End } V$ acts in the natural way on all the objects. In other words, since $df = -f \cdot \Phi$, it is clear that df will have the same symmetries as Φ .

For the case of the n -Lie algebras, we can therefore conclude the following:

Theorem 19. *Infinitesimal deformations of an n -Lie algebra are classified by $H^1(C^\bullet)$, where $C^\bullet \subset CL^\bullet(L; \text{End } V)$ is the subcomplex defined above Lemma 18. The obstructions to integrating an infinitesimal deformations live in $H^2(C^\bullet)$.*

We remark that whereas the natural map $H^1(C^\bullet) \rightarrow H^1(L; \text{End } V)$ is injective, the surjection $H^2(C^\bullet) \rightarrow HL^2(L; \text{End } V)$ may have kernel. Hence whereas $HL^1(L; \text{End } V) = 0$ is a sufficient (but not necessary) condition for V to be infinitesimally rigid, $HL^2(L; \text{End } V) = 0$ does not imply that infinitesimal deformations are unobstructed. In practice and in the absence of any strong structural results, these calculations are done by explicitly solving the cocycle and coboundary conditions in C^\bullet , whence the above subtleties will not play any rôle.

4.6. Deformations of metric n -Leibniz algebras. Many of the more physically interesting n -Leibniz algebras are metric — a concept defined in Section 3.2 — and in setting up a deformation theory one might wish to restrict the deformations to the class of metric n -Leibniz algebras. The data defining a metric n -Leibniz algebra consists of a vector space V with two additional structures, the n -bracket and the inner product, satisfying an open condition, namely the nondegeneracy of the inner product, and two algebraic conditions, namely the fundamental identity (29) and the compatibility condition (36) with the inner product.

A first, perhaps naive, approach to the deformation problem would be to deform both the n -bracket and the inner product. However we notice that we can always undo the deformation of the inner product via a change of basis: this is simply the fact that we can bring any inner product to a diagonal normal form with entries ± 1 . So it may be better to fix the inner product once and for all and deform the bracket in such a way that the compatibility condition (36) is preserved. That compatibility condition is equivalent to (37), which says that $D : L(V) \rightarrow \mathfrak{so}(V)$, where by $\mathfrak{so}(V)$ we mean the Lie algebra of skewsymmetric endomorphisms of V . This suggests restricting the deformation complex to a subspace $CL^\bullet(L; \mathfrak{so}(V)) \subset CL^\bullet(L; \text{End } V)$. To see that this is not obviously wrong, notice that an infinitesimal metric deformation $\varphi : L \rightarrow \mathfrak{so}(V)$ is trivial if $\varphi = df$, where $f \in \mathfrak{so}(V)$, so that we are only allowed an orthogonal change of basis. We have something to check, though.

Proposition 20. *$CL^\bullet(L; \mathfrak{so}(V))$ is a subcomplex of $CL^\bullet(L; \text{End } V)$.*

Proof. This follows from the fact that $\mathfrak{so}(V)$ is an L -subrepresentation of $\text{End } V$. The action of L on $\text{End } V$ is given by equation (46). If V is a metric n -Leibniz algebra, then $D(X) \in \mathfrak{so}(V)$ for all $X \in L$, whence if $\psi \in \mathfrak{so}(V)$, so are $[X, \psi] = [D(X), \psi]$, since $\mathfrak{so}(V)$ is a Lie subalgebra, and $[\psi, X] = [\psi, D(X)] - D(\psi \cdot X)$, for the same reasons. \square

A refinement of this result is the following

Proposition 21. *$CL^\bullet(L; \mathfrak{so}(V))$ is a graded Lie subalgebra of $CL^\bullet(L; \text{End } V)$.*

Proof. This is clear from the explicit expression given in [18] for the Lie bracket $[\alpha, \beta]$, where $\alpha \in CL^p(L; \mathfrak{so}(V))$ and $\beta \in CL^q(L; \mathfrak{so}(V))$. Applying this to $X_1 \otimes \cdots \otimes X_{p+q} \in L^{\otimes(p+q)}$, we see that it involves two kinds of terms: commutators in $\mathfrak{so}(V)$ or terms in the image of either α or β and hence both lie again in $\mathfrak{so}(V)$. We can see this explicitly in expression (72) for the case $p = q = 1$. \square

We remark that Proposition 21 implies Proposition 20, since the differential $d = [D, -]$ is the inner derivation defined by $D \in CL^1(L; \mathfrak{so}(V))$.

In summary we conclude with the metric version of Theorem 17.

Theorem 22. *Infinitesimal metric deformations of a metric n -Leibniz algebra V are classified by $HL^1(L; \mathfrak{so}(V))$ with $\mathfrak{so}(V)$ the representation of $L = V^{\otimes(n-1)}$ defined by (46). The obstructions to integrating such an infinitesimal deformation live in $HL^2(L; \mathfrak{so}(V))$.*

Finally, we may consider also deformations of a metric n -Lie algebra V . Now we have $L(V) = \Lambda^{n-1}V$ as explained in Section 4.5 and we restrict to the subrepresentation $\mathfrak{so}(V) \subset \text{End } V$. *Mutatis mutandis* we arrive at the following metric version of Theorem 19.

Theorem 23. *Infinitesimal metric deformations of a metric n -Lie algebra are classified by $H^1(C^\bullet)$, where $C^\bullet \subset CL^\bullet(L; \mathfrak{so}(V))$, for $L = \Lambda^{n-1}V$, is the subcomplex defined by*

$$C^p = CL^p(L; \mathfrak{so}(V)) \quad \text{for } p \neq 1, \text{ and} \quad (80)$$

$C^1 \subsetneq CL^1(L; \mathfrak{so}(V))$ consists of those maps $\varphi : L \rightarrow \mathfrak{so}(V)$ whose associated n -linear map given by equation (78) is totally skewsymmetric. The obstructions to integrating an infinitesimal deformations live in $H^2(C^\bullet)$.

Similar remarks to those in Section 4.5 after Theorem 19 apply here as well.

5. THE CASE $n = 3$

Due to their starring rôle in the construction of three-dimensional superconformal Chern–Simons theories with matter, 3-Lie algebras and more generally 3-Leibniz algebras deserve separate consideration.

5.1. The Leibniz algebra in the Faulkner construction. As shown in [4] all the metric 3-Leibniz algebras which have appeared in the construction of three-dimensional superconformal Chern–Simons theories with matter are special cases of a construction due originally to Faulkner [5]. We will recall this construction here and show how it too gives rise to a metric Leibniz algebra.

Let \mathfrak{g} be a real finite-dimensional Lie algebra with an ad-invariant symmetric bilinear form $(-, -)$ and let V be a finite-dimensional faithful representation of \mathfrak{g} with dual representation V^* . We will let $\langle -, - \rangle$ denote the dual pairing between V and V^* . Transposing the \mathfrak{g} -action defines for all $v \in V$ and $\alpha \in V^*$ an element $\mathcal{D}(v \otimes \alpha) \in \mathfrak{g}$ by

$$(X, \mathcal{D}(v \otimes \alpha)) = \langle X \cdot v, \alpha \rangle \quad \text{for all } X \in \mathfrak{g}, \quad (81)$$

where the \cdot indicates the \mathfrak{g} -action on V . Extending \mathcal{D} linearly, defines a \mathfrak{g} -equivariant map $\mathcal{D} : V \otimes V^* \rightarrow \mathfrak{g}$, which as shown in [4] is surjective because V is a faithful representation. To lighten the notation we will write $\mathcal{D}(v, \alpha)$ for $\mathcal{D}(v \otimes \alpha)$ in the sequel. The \mathfrak{g} -equivariance of \mathcal{D} is equivalent to

$$[\mathcal{D}(v, \alpha), \mathcal{D}(w, \beta)] = \mathcal{D}(\mathcal{D}(v, \alpha) \cdot w, \beta) + \mathcal{D}(w, \mathcal{D}(v, \alpha) \cdot \beta), \quad (82)$$

for all $v, w \in V$ and $\alpha, \beta \in V^*$, where the dual action $\mathcal{D}(v, \alpha) \cdot \beta$ is defined by

$$\langle w, \mathcal{D}(v, \alpha) \cdot \beta \rangle = -\langle \mathcal{D}(v, \alpha) \cdot w, \beta \rangle. \quad (83)$$

The map \mathcal{D} defines in turn a trilinear product

$$\begin{aligned} V \times V^* \times V &\rightarrow V \\ (v, \alpha, w) &\mapsto \mathcal{D}(v, \alpha) \cdot w. \end{aligned} \quad (84)$$

The fundamental identity (82) suggests defining a bracket on $V \otimes V^*$ by

$$[v \otimes \alpha, w \otimes \beta] = \mathcal{D}(v, \alpha) \cdot w \otimes \beta + w \otimes \mathcal{D}(v, \alpha) \cdot \beta, \quad (85)$$

which would make \mathcal{D} into a morphism. Indeed, we have the following

Proposition 24. *The bracket (85) turns $V \otimes V^*$ into a Leibniz algebra.*

Proof. We need only check the Leibniz identity (7):

$$[u \otimes \alpha, [v \otimes \beta, w \otimes \gamma]] - [[u \otimes \alpha, v \otimes \beta], w \otimes \gamma] - [v \otimes \beta, [u \otimes \alpha, w \otimes \gamma]] \stackrel{?}{=} 0. \quad (86)$$

We calculate each term in turn to obtain

$$\begin{aligned} [u \otimes \alpha, [v \otimes \beta, w \otimes \gamma]] &= [u \otimes \alpha, \mathcal{D}(v, \beta) \cdot w \otimes \gamma + w \otimes \mathcal{D}(v, \beta) \cdot \gamma] \\ &= \mathcal{D}(u, \alpha) \cdot \mathcal{D}(v, \beta) \cdot w \otimes \gamma + \mathcal{D}(v, \beta) \cdot w \otimes \mathcal{D}(u, \alpha) \cdot \gamma \\ &\quad + \mathcal{D}(u, \alpha) \cdot w \otimes \mathcal{D}(v, \beta) \cdot \gamma + w \otimes \mathcal{D}(u, \alpha) \cdot \mathcal{D}(v, \beta) \cdot \gamma, \\ [[u \otimes \alpha, v \otimes \beta], w \otimes \gamma] &= [\mathcal{D}(u, \alpha) \cdot v \otimes \beta + v \otimes \mathcal{D}(u, \alpha) \cdot \beta, w \otimes \gamma] \\ &= \mathcal{D}(\mathcal{D}(u, \alpha) \cdot v, \beta) \cdot w \otimes \gamma + w \otimes \mathcal{D}(\mathcal{D}(u, \alpha) \cdot v, \beta) \cdot \gamma \\ &\quad + \mathcal{D}(v, \mathcal{D}(u, \alpha) \cdot \beta) \cdot w \otimes \gamma + w \otimes \mathcal{D}(v, \mathcal{D}(u, \alpha) \cdot \beta) \cdot \gamma, \end{aligned}$$

and

$$\begin{aligned} [v \otimes \beta, [u \otimes \alpha, w \otimes \gamma]] &= [v \otimes \beta, \mathcal{D}(u, \alpha) \cdot w \otimes \gamma + w \otimes \mathcal{D}(u, \alpha) \cdot \gamma] \\ &= \mathcal{D}(v, \beta) \cdot \mathcal{D}(u, \alpha) \cdot w \otimes \gamma + \mathcal{D}(u, \alpha) \cdot w \otimes \mathcal{D}(v, \beta) \cdot \gamma \\ &\quad + \mathcal{D}(v, \beta) \cdot w \otimes \mathcal{D}(u, \alpha) \cdot \gamma + w \otimes \mathcal{D}(v, \beta) \cdot \mathcal{D}(u, \alpha) \cdot \gamma. \end{aligned}$$

Finally, putting it all together we find

$$\begin{aligned} [u \otimes \alpha, [v \otimes \beta, w \otimes \gamma]] - [[u \otimes \alpha, v \otimes \beta], w \otimes \gamma] - [v \otimes \beta, [u \otimes \alpha, w \otimes \gamma]] \\ = ([\mathcal{D}(u, \alpha), \mathcal{D}(v, \beta)] - \mathcal{D}(\mathcal{D}(u, \alpha) \cdot v, \beta) - \mathcal{D}(v, \mathcal{D}(u, \alpha) \cdot \beta)) \cdot w \otimes \gamma \\ + w \otimes ([\mathcal{D}(u, \alpha), \mathcal{D}(v, \beta)] - \mathcal{D}(\mathcal{D}(u, \alpha) \cdot v, \beta) - \mathcal{D}(v, \mathcal{D}(u, \alpha) \cdot \beta)) \cdot \gamma, \end{aligned}$$

which vanishes by virtue of the fundamental identity (82). \square

Therefore the bracket (85) defines a (left) Leibniz algebra structure on $V \otimes V^*$ making the map $\mathcal{D} : V \otimes V^* \rightarrow \mathfrak{g}$ into a Leibniz algebra morphism. Notice that $V \otimes V^* \cong \text{End } V$ as vector spaces, but the induced Leibniz algebra structure on $\text{End } V$ is different in general from the Lie algebra structure given by the commutator.

The vector space $V \otimes V^*$ has a natural \mathfrak{g} -invariant inner product induced from the dual pairing between V and V^* . Under the vector space isomorphism $V \otimes V^* \cong \text{End } V$, this inner product is simply the trace of the product of endomorphisms. On monomials, it is defined by

$$\langle v \otimes \alpha, w \otimes \beta \rangle = \langle w, \alpha \rangle \langle v, \beta \rangle, \quad (87)$$

for all $v, w \in V$ and $\alpha, \beta \in V^*$, and on all of $V \otimes V^*$ by extending linearly. Since this inner product is induced from the dual pairing, it is invariant under \mathfrak{g} , and hence under the left Leibniz action of $V \otimes V^*$ on itself.

Proposition 25. *The Leibniz algebra $V \otimes V^*$ with bracket defined by (85) is metric with respect to the inner product defined by (87).*

Proof. Let $X \in V \otimes V^*$ and let $\alpha, \beta \in V^*$ and $v, w \in V$. Then,

$$\begin{aligned} \langle [X, v \otimes \alpha], w \otimes \beta \rangle &= \langle \mathcal{D}(X) \cdot (v \otimes \alpha), w \otimes \beta \rangle \\ &= \langle \mathcal{D}(X) \cdot v \otimes \alpha + v \otimes \mathcal{D}(X) \cdot \alpha, w \otimes \beta \rangle \\ &= \langle \mathcal{D}(X) \cdot v, \beta \rangle \langle w, \alpha \rangle + \langle v, \beta \rangle \langle w, \mathcal{D}(X) \cdot \alpha \rangle \\ &= -\langle v, \mathcal{D}(X) \cdot \beta \rangle \langle w, \alpha \rangle - \langle v, \beta \rangle \langle \mathcal{D}(X) \cdot w, \alpha \rangle \\ &= -\langle v \otimes \alpha, \mathcal{D}(X) \cdot w \otimes \beta + w \otimes \mathcal{D}(X) \cdot \beta \rangle \\ &= -\langle v \otimes \alpha, \mathcal{D}(X) \cdot (w \otimes \beta) \rangle \end{aligned}$$

$$= -\langle v \otimes \alpha, [X, w \otimes \beta] \rangle .$$

□

5.2. 3-Leibniz algebras arising from the real Faulkner construction. A special case of the Faulkner construction recalled above is where V is a faithful unitary representation of \mathfrak{g} . This means that V is a real, complex or quaternionic representation of \mathfrak{g} possessing a \mathfrak{g} -invariant real symmetric, complex hermitian or quaternionic hermitian inner product, respectively. This gives rise, respectively, to a real orthogonal, complex unitary or quaternionic unitary representation of \mathfrak{g} . As explained in [19], we may take the real case as fundamental and think of the complex and quaternionic unitary cases as simply adding extra structure: an invariant orthogonal complex structure for the complex case and an anticommuting pair of such complex structures for the quaternionic case. As shown in [4], the real case corresponds precisely to the metric 3-Leibniz algebras constructed by Cherkis and Sämann in [20]. We briefly recall this construction here in order to later set up the deformation theory of such algebras.

We will first briefly review the case of $(V, \langle -, - \rangle)$ a real inner product space admitting a faithful orthogonal action of a real metric Lie algebra \mathfrak{g} . The inner product on V sets up an isomorphism $\flat : V \rightarrow V^*$ of \mathfrak{g} -modules, defined by $v^\flat = \langle v, - \rangle$, with inverse $\sharp : V^* \rightarrow V$. The map $\mathcal{D} : V \otimes V^* \rightarrow \mathfrak{g}$ defined by equation (81) induces a map $D : V \otimes V \rightarrow \mathfrak{g}$, by $D(v \otimes w) = \mathcal{D}(v \otimes w^\flat)$. In other words, for all $v, w \in V$ and $X \in \mathfrak{g}$, we have

$$(D(v \otimes w), X) = \langle X \cdot v, w \rangle . \quad (88)$$

It follows from the \mathfrak{g} -invariance of the inner product that

$$(D(v \otimes w), X) = \langle X \cdot v, w \rangle = -\langle X \cdot w, v \rangle = -(D(w \otimes v), X) , \quad (89)$$

whence

$$D(v \otimes w) = -D(w \otimes v) . \quad (90)$$

This means that D factors through a map also denoted $D : \Lambda^2 V \rightarrow \mathfrak{g}$.

Using D we can define a 3-bracket on V by

$$[u, v, w] := D(u \wedge v) \cdot w , \quad (91)$$

for all $u, v, w \in V$. The resulting 3-Leibniz algebra, which appeared originally in [5] but more recently in [20] in the context of superconformal Chern–Simons-matter theories, satisfies the following axioms for all $x, y, z, v, w \in V$:

(a) the *orthogonality* condition

$$\langle [x, y, z], w \rangle = -\langle z, [x, y, w] \rangle ; \quad (92)$$

(b) the *symmetry* condition

$$\langle [x, y, z], w \rangle = \langle [z, w, x], y \rangle ; \quad (93)$$

(c) and the *fundamental identity*

$$[x, y, [v, w, z]] - [v, w, [x, y, z]] = [[x, y, v], w, z] + [v, [x, y, w], z] . \quad (94)$$

It follows from the orthogonality and symmetry conditions that $[x, y, z] = -[y, x, z]$ for all $x, y, z \in W$, which is nothing but equation (90). We will call such metric 3-Leibniz algebras **Cherkis–Sämann 3-algebras**. They have as a special case the metric 3-Lie algebras appearing in the maximally supersymmetric $N = 8$ theory of Bagger–Lambert [1, 3] and Gustavsson [2], wherein the 3-bracket is totally skewsymmetric. Another special case of these 3-Leibniz algebras corresponds to metric Lie triple systems, for which the 3-bracket obeys $[x, y, z] + [y, z, x] + [z, x, y] = 0$. Metric Lie triple systems are characterised by the fact that they embed

into $\mathfrak{g} \oplus V$ as a real metric \mathbb{Z}_2 -graded Lie algebra and are in one-to-one correspondence with pseudoriemannian symmetric spaces.

An easy consequence of the results in Section 5.1 is that the Leibniz algebra $L(V) = \Lambda^2 V$ is metric relative to the standard determinantal inner product:

$$\langle u \wedge v, w \wedge z \rangle = \langle u, w \rangle \langle v, z \rangle - \langle u, z \rangle \langle v, w \rangle . \quad (95)$$

5.3. Deformation complex of Cherkis–Sämman 3-algebras. Let us now set up the deformation theory of these metric 3-Leibniz algebras, obtained via the Faulkner construction associated to a real orthogonal representation of a metric Lie algebra. As discussed above, such an algebra consists of a real vector space V and a linear map $D : \Lambda^2 V \rightarrow \mathfrak{so}(V)$ satisfying the fundamental identity (3) and, in addition, the symmetry condition (93). In the absence of the symmetry condition, the deformation theory of those algebras are covered by the results in Section 4.6 and in particular by Theorem 22, but applied to the Leibniz algebra $\Lambda^2 V$. The symmetry condition requires special consideration. The situation here is analogous to that of metric n -Lie algebras, except that instead of total skewsymmetry of the bracket, the additional algebraic condition we are imposing is equation (93). Following the discussion in Section 4.5, we define a graded subspace $C^\bullet \subset CL^\bullet(L; \mathfrak{so}(V))$, where $L = \Lambda^2 V$, by $C^p = CL^p(L; \mathfrak{so}(V))$ for $p \neq 1$ and $C^1 \subsetneq CL^1(L; \mathfrak{so}(V))$ consists of those $\varphi : \Lambda^2 V \rightarrow \mathfrak{so}(V)$ such that

$$\langle \varphi(u \wedge v) \cdot x, y \rangle = \langle \varphi(x \wedge y) \cdot u, v \rangle , \quad (96)$$

for all $u, v, x, y \in V$.

Lemma 26. *The subspace C^\bullet so defined is a subcomplex of $CL^\bullet(L; \mathfrak{so}(V))$.*

Proof. We need only verify that the image of the differential $d : CL^0(L; \mathfrak{so}(V)) \rightarrow CL^1(L; \mathfrak{so}(V))$ actually lives inside C^1 . To this end let $f \in \mathfrak{so}V = CL^0(L; \mathfrak{so}(V))$. Let $X \in L$ and $\psi \in \mathfrak{so}(V) = CL^0(L; \mathfrak{so}(V))$ and consider $d\psi \in CL^1(L; \mathfrak{so}(V))$. Then

$$d\psi(X) = -[\psi, X] = -[\psi, D(X)] + D(\psi \cdot X) , \quad (97)$$

whence

$$\begin{aligned} \langle d\psi(u \wedge v) \cdot x, y \rangle &= -\langle [\psi, D(u \wedge v)] \cdot x, y \rangle + \langle D(\psi \cdot (u \wedge v)) \cdot x, y \rangle \\ &= -\langle \psi \cdot D(u \wedge v) \cdot x, y \rangle + \langle D(u \wedge v) \cdot \psi \cdot x, y \rangle \\ &\quad + \langle D(\psi \cdot u \wedge v) \cdot x, y \rangle + \langle D(u \wedge \psi \cdot v) \cdot x, y \rangle \\ &= \langle D(u \wedge v) \cdot x, \psi \cdot y \rangle + \langle D(u \wedge v) \cdot \psi \cdot x, y \rangle \\ &\quad + \langle D(\psi \cdot u \wedge v) \cdot x, y \rangle + \langle D(u \wedge \psi \cdot v) \cdot x, y \rangle \\ &= \langle D(x \wedge \psi \cdot y) \cdot u, v \rangle + \langle D(\psi \cdot x \wedge y) \cdot u, v \rangle \\ &\quad + \langle D(x \wedge y) \cdot \psi \cdot u, v \rangle + \langle D(x \wedge y) \cdot u, \psi \cdot v \rangle \\ &= \langle D(\psi \cdot (x \wedge y)) \cdot u, v \rangle + \langle [D(x \wedge y), \psi] \cdot u, v \rangle \\ &= \langle D(\psi \cdot (x \wedge y)) \cdot u, v \rangle - \langle [\psi, D(x \wedge y)] \cdot u, v \rangle \\ &= \langle d\psi(x \wedge y) \cdot u, v \rangle , \end{aligned}$$

whence $d\psi \in C^1$. □

In complete analogy to Theorem 19, we have the following

Theorem 27. *Infinitesimal deformations of a metric 3-Leibniz algebra V obtained by the real Faulkner construction are classified by $H^1(C^\bullet)$, where $C^\bullet \subset CL^\bullet(L; \text{End } V)$ is the subcomplex defined above. The obstructions to integrating an infinitesimal deformations live in $H^2(C^\bullet)$.*

Similar remarks to those after Theorem 19 apply here as well.

5.4. Some calculations. Since, in the absence of general theoretical results on the cohomology of Leibniz algebras, calculations of deformations (or rigidity) of metric 3-Leibniz algebras seem to be amenable only to explicit solution of the cocycle and coboundary equations, one is inevitably, albeit reluctantly, driven to work relative to a basis in such a way that one can then harness the power of symbolic computation.

Let V be an N -dimensional 3-Leibniz algebra with basis (e_a) . Relative to this basis, the 3-bracket is given by the structure constants F_{abc}^d defined by

$$[e_a, e_b, e_c] = F_{abc}^d e_d, \quad (98)$$

where here and in the sequel we will employ the summation convention.

The corresponding Leibniz algebra $L = V \otimes V$ is N^2 -dimensional and has basis $e_{ab} := e_a \otimes e_b$. A 0-cochain $f \in CL^0(L; \text{End } V) = \text{End } V$ is given by a tensor f_a^b defined by

$$f(e_b) = f_a^b e_b, \quad (99)$$

whereas a 1-cochain $\varphi \in CL^1(L; \text{End } V)$ is given by a tensor φ_{abc}^d defined by

$$\varphi(e_{ab})(e_c) = \varphi_{abc}^d e_d. \quad (100)$$

Such a 1-cochain is 1-coboundary, $\varphi = df$, if and only if

$$\varphi_{abc}^d = f_a^e F_{ebc}^d + f_b^e F_{aec}^d + f_c^e F_{abe}^d - F_{abc}^e f_e^d, \quad (101)$$

where it is a 1-cocycle, $d\varphi = 0$, if and only if

$$\varphi_{cde}^g F_{abg}^f - F_{abe}^g \varphi_{cdg}^f + F_{cde}^g \varphi_{abg}^f - \varphi_{abe}^g F_{cdg}^f - \varphi_{abd}^g F_{cge}^f - \varphi_{abc}^g F_{gde}^f - F_{abc}^g \varphi_{gde}^f - F_{abd}^g \varphi_{cge}^f = 0. \quad (102)$$

Equations (101) and (102) are not altered when we consider 3-Lie algebras, except that now φ_{abc}^d is totally skewsymmetric in abc . As we remarked in more generality after the proof of Lemma 18, we see here explicitly that the coboundary df is simply the action of the endomorphism $f \in \mathfrak{gl}(V)$ on the tensor F , and hence if F belongs to some submodule of $\mathfrak{gl}(V)$, so will df .

This has the following practical upshot for the computation of the infinitesimal deformations. To deform in a class of algebras larger than the one the original algebra lies in, e.g., to deform a 3-Lie algebra as a 3-Leibniz algebra, one simply relaxes the total skewsymmetry of φ_{abc}^d from the start. The space of coboundaries will not change, but the space of cocycles might be enlarged, as one would expect.

To illustrate this, let us consider the unique simple euclidean 3-Lie algebra, here denoted S_4 .

Example 28 (The Leibniz algebra of the simple euclidean 3-Lie algebra S_4). Take $V = \mathbb{R}^4$ with the standard inner product. Let (e_a) , for $1 \leq a \leq 4$, be an orthonormal basis. The associated Leibniz algebra is $\Lambda^2 \mathbb{R}^4$ with basis $(e_{ab} := e_a \wedge e_b)$, for $1 \leq a < b \leq 4$. The bracket of the 3-Lie algebra is given by

$$[e_a, e_b, e_c] = \varepsilon_{abcd} e_d, \quad (103)$$

with the conventions that $\varepsilon_{1234} = +1$. The bracket in the Leibniz algebra is given by

$$[e_{ab}, e_{cd}] = \varepsilon_{abce} e_{ed} + \varepsilon_{abde} e_{ce}. \quad (104)$$

The Lie bracket in $\mathfrak{g} = \mathfrak{so}(4)$ is given by

$$[D(e_{ab}), D(e_{cd})] = \varepsilon_{abce} D(e_{ed}) + \varepsilon_{abde} D(e_{ce}). \quad (105)$$

Since D has no kernel, it is an isomorphism of Leibniz algebras $\Lambda^2 \mathbb{R}^4 \rightarrow \mathfrak{so}(4)$. Since $\mathfrak{so}(4)$ is Lie, so is $\Lambda^2 \mathbb{R}^4$. The inner product in V is such that the e_a are orthonormal, and this implies that in the Leibniz algebra

$$\langle e_{ab}, e_{cd} \rangle = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}, \quad (106)$$

whereas in the Lie algebra of inner derivations

$$\langle D(e_{ab}), D(e_{cd}) \rangle = \varepsilon_{abcd} . \quad (107)$$

An explicit calculation (made less painful using symbolic computation, e.g., Mathematica) reveals that S_4 is rigid as a 3-Lie algebra, whereas it admits a one-parameter deformation as a 3-Leibniz algebra

$$[e_a, e_b, e_c]_t = \varepsilon_{abcd} e_d + t(\delta_{bc} e_a - \delta_{ac} e_b) . \quad (108)$$

It is interesting that this deformed 3-Leibniz algebra is already of Faulkner type. As such this deformation can be understood from the Faulkner construction, as we now show.

For $t^2 \neq 1$, the Faulkner Lie algebra is $\mathfrak{g} \cong \mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ which admits, up to rescalings, a pencil of \mathfrak{g} -invariant inner products on \mathfrak{g} . The point $t = 0$ corresponds to the initial point in the deformation, namely the 3-Lie algebra S_4 and corresponds as well to an inner product on \mathfrak{g} which has split signature. For $t^2 > 1$ the signature of the inner product is either positive-definite (for $t < -1$) or negative-definite ($t > 1$). As $t \rightarrow \pm\infty$ the algebra tends to the metric Lie triple system associated to $S^4 = \text{SO}(5)/\text{SO}(4)$, thought of as a riemannian symmetric space. At the points $t = \pm 1$, the Faulkner Lie algebra is isomorphic to $\mathfrak{so}(3)$: the selfdual $\mathfrak{so}(3)$ for $t = -1$ and the antiselfdual for $t = +1$. The inner product in either case is a multiple of the Killing form: being positive-definite for $t = -1$ and negative-definite for $t = +1$.

5.5. Deformations of the Faulkner data. The above example suggests that we ought to be able to understand deformations of the metric 3-algebra in terms of deformations of its Faulkner data. Let us consider a metric 3-algebra of Faulkner type on a finite-dimensional real vector space V with symmetric inner product $\langle -, - \rangle$. The Faulkner data is given by a metric Lie algebra \mathfrak{g} with ad-invariant inner product $\kappa := (-, -)$ and an embedding $\iota : \mathfrak{g} \rightarrow \mathfrak{so}(V)$. Sylvester's Law of Inertia says that the signature of a nondegenerate inner product cannot change under deformations, whence we may take the inner product on V and hence $\mathfrak{so}(V)$ to be rigid. This means that the structures getting deformed are the Lie bracket on \mathfrak{g} , the inner product κ on \mathfrak{g} and the embedding ι . In the above example we see that there are values of the deformation parameter where \mathfrak{g} drops dimension. In order to take this into account, it is convenient to fix the underlying vector space of \mathfrak{g} but allow κ to be degenerate and ι to have nontrivial kernel. Then the true Faulkner Lie algebra is not \mathfrak{g} but $\mathfrak{g}/\text{rad } \kappa$, where the radical

$$\text{rad } \kappa = \{X \in \mathfrak{g} \mid \kappa(X, Y) = 0 \ \forall Y \in \mathfrak{g}\} \quad (109)$$

of κ is an ideal of \mathfrak{g} , whence the quotient $\mathfrak{g}/\text{rad } \kappa$ is a metric Lie algebra. Indeed, it follows from equation (88) that $D(x \otimes y)$ is only defined modulo $\text{rad } \kappa$ and that if $X \in \text{rad } \kappa$ then $X \in \ker \iota$.

By a **deformation of the Faulkner data** we mean a one-parameter family consisting, for every t in a neighbourhood of 0, of

- a linear map $[-, -]_t : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$ subject to the Jacobi identity

$$[X, [Y, Z]_t]_t = [[X, Y]_t, Z]_t + [Y, [X, Z]_t]_t , \quad (110)$$

- a bilinear form $\kappa_t : S^2 \mathfrak{g} \rightarrow \mathbb{R}$ subject to the ad-invariance condition

$$\kappa_t([X, Y]_t, Z) = -\kappa_t(Y, [X, Z]_t) , \quad (111)$$

- and a morphism $\iota_t : \mathfrak{g} \rightarrow \mathfrak{so}(V)$, whence subject to

$$\iota_t[X, Y]_t = [\iota_t X, \iota_t Y] , \quad (112)$$

where the bracket on the right-hand side is the one in $\mathfrak{so}(V)$.

Associated to this data there is a map $D_t : \Lambda^2 V \rightarrow \mathfrak{g}$ defined by

$$\kappa_t(D_t(x \wedge y), X) = \langle \iota_t X \cdot x, y \rangle , \quad (113)$$

for all $X \in \mathfrak{g}$ and $x, y \in V$, and a corresponding 3-bracket

$$[x, y, z]_t = \iota_t D_t(x \wedge y) \cdot z . \quad (114)$$

Notice that $D_t(x \wedge y)$ is only defined up to $\text{rad } \kappa_t$ and that if $X \in \text{rad } \kappa_t$, then $\iota_t X = 0$, whence the Faulkner Lie algebra is $\mathfrak{g}_t / \text{rad } \kappa_t$ and ι_t factors through an embedding $\mathfrak{g}_t / \text{rad } \kappa_t \rightarrow \mathfrak{so}(V)$.

The deformation equations (110), (111) and (112) are quadratic. Linearising them around $t = 0$ we obtain the linear equations which define an infinitesimal deformation of the Faulkner data. Let us write

$$\begin{aligned} [X, Y]_t &= [X, Y] + \sum_{k \geq 1} t^k \varphi_k(X, Y) \\ \kappa_t(X, Y) &= \kappa(X, Y) + \sum_{k \geq 1} t^k \mu_k(X, Y) \\ \iota_t(X) &= \iota(X) + \sum_{k \geq 1} t^k \lambda_k(X) , \end{aligned} \quad (115)$$

in terms of which, the infinitesimal deformations are given by

$$\begin{aligned} \varphi(X, [Y, Z]) + [X, \varphi(Y, Z)] - \varphi([X, Y], Z) - \varphi(Y, [X, Z]) - [\varphi(X, Y), Z] - [Y, \varphi(X, Z)] &= 0 \\ \kappa(\varphi(X, Y), Z) + \mu([X, Y], Z) + \kappa(Y, \varphi(X, Z)) + \mu(Y, [X, Z]) &= 0 \\ \iota(\varphi(X, Y)) + \lambda([X, Y]) - [\iota(X), \lambda(Y)] - [\lambda(X), \iota(Y)] &= 0 , \end{aligned} \quad (116)$$

where $\varphi := \varphi_1$, $\mu := \mu_1$ and $\lambda := \lambda_1$. The first equation is simply the cocycle condition for $\varphi \in C^2(\mathfrak{g}; \mathfrak{g})$.

A deformation is trivial if it is the result of the action of a one-parameter subgroup of the general linear group $\text{GL}(\mathfrak{g})$. For an infinitesimal deformation this means that

$$\begin{aligned} \varphi(X, Y) &= \psi([X, Y]) - [\psi(X), Y] - [X, \psi(Y)] \\ \mu(X, Y) &= -\kappa(\psi(X), Y) - \kappa(X, \psi(Y)) \\ \lambda(X) &= -\iota(\psi(X)) , \end{aligned} \quad (117)$$

for some $\psi \in \mathfrak{gl}(\mathfrak{g})$. The first equation simply says that $\varphi = -d\psi$ for $\psi \in C^1(\mathfrak{g}; \mathfrak{g})$. One can easily check that if φ , μ and λ are given as in equations (117), then they also satisfy equations (116).

It is not clear, however, that computing deformations of the Faulkner data is any easier than computing deformations of the 3-Leibniz algebra itself; although if \mathfrak{g} is semisimple, then one can do better. It may seem that this is a very special case, but notice that if V is positive-definite, then $\mathfrak{g} < \mathfrak{so}(V)$ is reductive, whence the direct sum of a semisimple and an abelian Lie algebras. So taking \mathfrak{g} semisimple is an important special case.

Theorem 29. *Let $(\mathfrak{g}, \kappa, \iota)$ be Faulkner data for a Cherkis–Sämann 3-algebra V , where \mathfrak{g} is semisimple. Then only κ deforms and does so by rescaling the Killing form in each of its simple ideals.*

Proof. Indeed, if \mathfrak{g} is semisimple, it is rigid and hence one can assume that $[-, -]_t$ is constant and equal to the original bracket. The notion of trivial deformation now changes, of course, since $\text{GL}(\mathfrak{g})$ does not act on \mathfrak{g} via automorphisms. A trivial deformation is one which corresponds to the action of a one-parameter subgroup of $\text{Aut}(\mathfrak{g})$. If \mathfrak{g} is semisimple, this means a one-parameter subgroup of the adjoint group.

Let $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$ denote the decomposition of \mathfrak{g} into its simple ideals. Any ad-invariant symmetric bilinear form on \mathfrak{g} takes the form

$$\kappa = r_1 \kappa_1 + \cdots + r_k \kappa_k ,$$

where $r_i \in \mathbb{R}$ and κ_i is the Killing form on \mathfrak{s}_i . Now the identity component of the adjoint group of \mathfrak{g} (where one-parameter subgroups live) is the direct product of the identity components of the adjoint groups of each of the \mathfrak{s}_i . These preserve the κ_i , whence trivial deformations actually leave κ invariant. The deformations of κ consist of changing the r_i , so that

$$\kappa_t = r_1(t) \kappa_1 + \cdots + r_k(t) \kappa_k .$$

Finally we show that ι is also rigid. Indeed as shown in [21], infinitesimal deformations of the morphism $\iota : \mathfrak{g} \rightarrow \mathfrak{so}(V)$ are classified by the Lie algebra cohomology space $H^1(\mathfrak{g}; \mathfrak{so}(V))$, where $\mathfrak{so}(V)$ becomes a \mathfrak{g} -module via $[\iota(X), -]$ for $X \in \mathfrak{g}$. However for \mathfrak{g} semisimple, $H^1(\mathfrak{g}, \mathfrak{M}) = 0$ for any \mathfrak{g} -module \mathfrak{M} by the Whitehead Lemma. Therefore ι is rigid. \square

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